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Geometry and Combinatorics of Julia Sets of Real Quadratic Maps

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For real λ a correspondence is made between the Julia set B_λ for $z \to (z-\lambda)^2$, in the hyperbolic case, and the set of λ-chains $\{\lambda \pm \sqrt{(\lambda \pm \sqrt{(\lambda \pm \cdots\}}}$, with the aid of Cremer's theorem. It is shown how a number of features of B_λ can be understood in terms of λ-chains. The structure of B_λ is determined by certain equivalence classes of λ-chains, fixed by orders of visitation of certain real cycles; and the bifurcation history of a given cycle can be conveniently computed via the combinatorics of λ-chains. The functional equations obeyed by attractive cycles are investigated, and their relation to λ-chains is given. The first cascade of period-doubling bifurcations is described from the point of view of the associated Julia sets and λ-chains. Certain "Julia sets" associated with the Feigenbaum function and some theorems of Lanford are discussed.

KEY WORDS: Iterated maps; Julia sets; cascades of bifurcations; Feigenbaum functional equation; universal scaling.

1. INTRODUCTION

Iterated quadratic maps have played a central role in the continuing formulation of the general theory of iterated maps on intervals(14,25,29,35,36,39) and quantitative universality.(16,17,23,30) Studies of one-parameter families of quadratic maps suggest results which hold more generally. Fundamental to the behavior of iterates under a quadratic map is the structure of the associated Julia set; for example, the development of chaotic behavior for iterated real quadratic maps is associated with the arrival in the real line of successively more of the Julia set as a parameter is

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varied; and cascades of bifurcations are related to sequences of topological structural changes in the Julia set.

There are other reasons for interest in the structure of Julia sets of quadratic maps in particular. Features, such as certain "universal shapes," associated with the Julia sets of quadratic maps occur much more generally in connection with other polynomial-like maps.[15,20] The theory of balanced measures on Julia sets[2,6,8,10,11] has led to the discovery of certain tridiagonal matrix operators with almost periodic structure, whose spectra are Julia sets of polynomial maps.[1,5,9] The zeros of the partition function for certain hierarchical Ising models accumulate on Julia sets.[18] The spectrum and spectral density of a model Schrödinger equation on a Sierpinski gasket,[19,37] of interest in percolation theory and also because of its renormalization properties, is a condensed Julia set and is closely related to the Julia set of a real quadratic map.[7]

In this paper, we examine in detail the family of Julia sets B_λ for the quadratic maps $T_\lambda(z) = (z - \lambda)^2$, where λ is a real parameter. We are concerned mainly with the hyperbolic cases ($-\infty < \lambda < -1/4$; $-1/4 < \lambda < 2$ and T_λ possesses an attractive cycle; $2 < \lambda < \infty$); and particularly with the description of B_λ in terms of λ-chains[4,38]

$$\lambda \pm \sqrt{(\lambda \pm \sqrt{(\lambda \pm \cdots))}}$$

We consider certain combinatorics of λ-chains and show how these are related both to the calculus of itineraries[14,25,29,35,36] and to the topology of B_λ. We also describe the first cascade of period-doubling bifurcations from the point of view of the associated Julia sets and λ-chains. Finally, we discuss certain "Julia sets" associated with the Cvitanovic–Feigenbaum–Coullet–Tresser functional equation; this involves a number of conjectures and is motivated in part by some theorems of Lanford.[30]

2. LAMBDA-CHAINS AND THE JULIA SET

2.1. The Julia Set

Let $\mathbb{C}$ be the complex plane and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let B_λ denote the Julia set[13,21,27,28] for the mapping $T_\lambda\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by

$$T_\lambda(z) = (z - \lambda)^2$$

where $\lambda \in \mathbb{R}$. We will use the notation $T_\lambda^0(z) = z$ and $T_\lambda^n(z)$ for $T_\lambda(z)$ composed with itself n times. B_λ can be described with the aid of the following definition.

Definition 1. If a finite set of distinct points $\{z_1, z_2, \ldots, z_k\} \subset \mathbb{C}$ is such that $T_\lambda(z_1) = z_2$, $T_\lambda(z_2) = z_3, \ldots, T_\lambda(z_k) = z_1$; then $\{z_1, z_2, \ldots, z_k\}$ is a *k-cycle* of T_λ. The k-cycle is *attractive*, *indifferent*, or *repulsive*, according as

$$\left| \frac{d}{dz} T_\lambda^k(z)|_{z_1} \right|$$

is strictly less than unity, equal to unity, or strictly greater than unity, respectively.

B_λ is the closure of the set of repulsive k-cycles of T_λ, for all finite k.(27) An important characterization is that B_λ is the set of points $z \in \mathbb{C}$, where $\{T^n(z)\}_{n=0}^{\infty}$ is not a normal family.(13) B_λ can also be described in terms of infinite compositions of branches of the inverse of T_λ, which is the point of view we adopt. This description is developed next.

2.2. Cremer's Theorem for $T_\lambda(z)$

We begin by describing the Riemann surfaces, and convenient branch cut structures, for the inverses of $T_\lambda^n(z)$, $n = 1, 2, 3, \ldots$

Let R denote the inverse of T_λ. Then the branch points of R are 0 and ∞. Let γ denote any simple continuous path which connects 0 and ∞ on the Riemann sphere $\hat{\mathbb{C}}$. Let $\mathbb{D}$ denote two copies of the Riemann sphere $\hat{\mathbb{C}}$, each slit along the path γ, and joined one to the other at the lips of the slit; see Fig. 1. One seam, which we continue to call γ, belongs to one of the spheres,

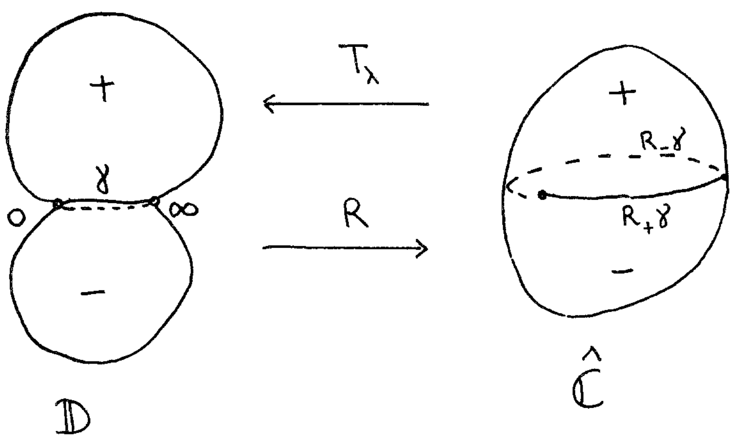


Fig. 1. This illustrates the conformal equivalence between $\mathbb{D}$ and $\hat{\mathbb{C}}$ provided by R. $\mathbb{D}$ consists of two spheres slit along γ and joined there. The points 0 and ∞ occur only once, but $\gamma - \{0, \infty\}$ appears twice, once on each sphere. $\hat{\mathbb{C}}$ is divided into two components, labeled + and −, by the simple Jordan curve $R_+\gamma \cup R_-\gamma$.

while the other, which we call γ', belongs to the other sphere. The end points 0 and ∞ of γ (and γ') appear only once, being common to both spheres. R maps $\mathbb{D}$ one-to-one onto $\hat{\mathbb{C}}$, and we use the notation R_+ for this mapping restricted to one of the spheres, and R_- for the restriction to the other sphere. The domain of each mapping can be taken to be $\hat{\mathbb{C}}$, and their ranges divide $\hat{\mathbb{C}}$ into two components separated by the Jordan curve $R_+\gamma \cup R_-\gamma$. R_- is the analytic continuation of R_+ across $\gamma - \{0, \infty\}$ and vice versa.

Using R_+ and R_- we can build up chains of inverse maps, such as $R_+(R_-(R_-(R_+(R_-(z)))))$, which represent branches of the inverse mappings $R^n(z)$, $n \in \{1, 2, 3,...\}$. The domain of $R^n(z)$ is $\mathbb{D}^n$, which consists of 2^n copies of $\hat{\mathbb{C}}$ slit and interconnected along paths belonging to the set $\{\gamma, T_\lambda \gamma,..., T_\lambda^{n-1}\gamma\}$. $T_\lambda^{n-1}\gamma$ appears on two of the spheres, $T_\lambda^{n-2}\gamma$ on four of the spheres,..., and γ on all of them. The finite critical points are: $T_\lambda^{n-1}0$, which occurs once, common to the two spheres which are slit and joined along $T_\lambda^{n-1}\gamma$; $T_\lambda^{n-2}0$, which occurs twice, once common to one pair of spheres which are slit and joined along $T_\lambda^{n-2}\gamma$ and once common to the other pair of spheres which are slit and joined along the same arc; ...; and 0, which occurs 2^{n-1} times being common on each of the 2^{n-1} pairs of spheres which are connected along γ. The point at infinity is common to all of the spheres. The lips of the slits, along which the spheres are joined, are the branch cuts of R^n. The domain of a single branch of $R^n(z)$ consists of the projection from $\mathbb{D}^n$ onto $\hat{\mathbb{C}}$ of one of the 2^n copies of $\hat{\mathbb{C}}$, complete with the branch cuts which belong to it. We refer to this domain as the original sheet of the domain of the given branch of $R^n(z)$. By analytically continuing a branch of R^n from its original sheet across a branch cut on that sheet, one arrives at another branch of R^n, complete with its own collection of branch points and cuts. Hence, a branch of $R^n(z)$ can be defined as a holomorphic function on a domain which extends from the original sheet onto other sheets by crossing available cuts.

To represent all of the branches of $R^n(z)$, let Ω denote the set of all half-infinite chains of $+1$ and -1, so that $\omega \in \Omega$ if and only if

$$\omega = (e_1, e_2, e_3,...)$$

where each $e_i \in \{-1, 1\}$. Then we write

$$R_\omega^n(z) = R_{s(e_1)}(R_{s(e_2)}(... R_{s(e_n)}(z)...))$$

where $s(+1) = +$ and $s(-1) = -$. We denote the set of all finite branch points of all inverse branches of T_λ by

$$C = \{T_\lambda^n(0) \mid n \in \{0, 1, 2,...\}\}$$

Then every one of the functions $R^n_\omega(z)$ is meromorphic in any simply connected domain D of $\mathbb{D}$ such that $P(D) \cap \bar{C} = \phi$, where $P\colon \mathbb{D} \to \hat{\mathbb{C}}$ denotes the projection which identifies elements of $\mathbb{D}$ with the corresponding points in $\hat{\mathbb{C}}$.

The following theorem is readily deduced from Ref. 13 (p. 113, Theorem 6.2, and Lemma 6.3).

Theorem (due to Cremer 1932). Let $\{R^{n_i}_{\omega_i}(z)\}$ denote any infinite set of finite compositions of inverse branches of T_λ, and let D be any simply connected domain on $\mathbb{D}$ such that $P(D) \cap \bar{C} = \phi$, and such that $P(D)$ contains no accumulation point of successors of a point outside B_λ. Then $\{R^{n_i}_{\omega_i}(z)\}$ is normal in D and every convergent subsequence of

$$\{R^{n_i}_{\omega_i}(z)\}$$

tends to a constant which belongs to B_λ. Moreover, if $b \in B_\lambda$ then there is a sequence of $\{R^{m_j}_{\omega_j}(z)\}$ which converges to b uniformly on closed subsets of D.

The above theorem tells us we can set up a correspondence between the points of B_λ and infinite sequences of inverse maps. To make such a correspondence we must first specify γ. Since the branch points of R^n are all real for λ real, it is convenient to choose γ to be either the positive real axis or the negative real axis.

Definition 2. When γ is the positive real axis, define

$$R_\pm(z) = \lambda \pm \sqrt{r}\, e^{i\theta/2}$$

where $z = re^{i\theta}$ with $r \geqslant 0$ and $0 \leqslant \theta < 2\pi$. When γ is the negative real axis, define

$$\tilde{R}_\pm(z) = \lambda \pm \sqrt{r}\, e^{i\theta/2}$$

where $z = re^{i\theta}$ with $r \geqslant 0$ and $-\pi < \theta \leqslant \pi$.

Defintion 3. Let $\omega \in \Omega$. Let γ be the positive real axis and let $b(\omega)$ denote the set of all limit points of $\{R^n_\omega(z)\colon n = 1, 2, \ldots, \infty\}$, for some specified choice of z. The set valued function $b\colon \Omega \to \{\text{subsets of } \hat{\mathbb{C}}\}$ is called a positive axis λ-chain function, and the set $b(\omega)$ is called a positive axis *λ-chain*. Similarly, define $\tilde{b}$ to be a negative axis λ-chain function, and $\tilde{b}(\omega)$ to be a *negative axis λ-chain*, by taking γ to be the negative real axis.

The following theorems, based on Cremer's theorem, give conditions under which either b or $\tilde{b}$ is a single-valued mapping from Ω onto B_λ. Each theorem corresponds to a different range of real λ-values; and suitable choices for z, required in Definition 3, are implied.

Definition 4. Let $\{z_1, z_2, \ldots, z_k\}$ be an attractive k-cycle of T_λ. Then the *attractive set* of the k-cycle is defined to be A_k, where

$$A_k = \{z \in \mathbb{C} \mid \lim_{n\to\infty} T_\lambda^{nk}(z) \in \{z_1, z_2, \ldots, z_k\}\}$$

We remark that T_λ possesses at most one attractive k-cycle because R has only one finite branch point, and there must be at least one such point in the attractive set of a k-cycle. In particular, attractive k-cycles are always real, which is one reason why real axis cuts are most convenient.

Theorem 1. Let γ be the positive axis branch cut. Let $\lambda \in [-1/4, 2]$ be such that T_λ admits an attractive k-cycle $\{z_1, z_2, \ldots, z_k\}$. Let Q be any region in the complement of the attractive set, with $\infty \cap Q = \phi$. Then for each $\omega \in \Omega$, $\{R_\omega^n(z)\}_{n=1}^\infty$ converges uniformly on closed subsets of Q to a single element $b(\omega) \in B_\lambda$. Moreover the mapping b: $\Omega \to B_\lambda$ is onto.

Proof. Since the branch point λ is attracted to the k-cycle, $\bar{C}$ can be covered by a finite union of disjoint convex open sets F, such that $\bar{F} \subset A_k$. Then $(\mathbb{C} - \bar{F}) \cap \gamma$ is a finite union of disjoint connected components. Let N_i be an open neighborhood of the ith component, such that $N_i \cap N_j = \phi$ for $i \neq j$ and $\bigcup_i N_i \cap \gamma = (\mathbb{C} - \bar{F}) \cap \gamma$, (‡). Then we define a domain D on $\mathbb{D}$ to consist of $\mathbb{C} - \bar{F}$ lifted to one of the spheres of $\mathbb{D}$, together with $\bigcup_i N_i$ lifted to the other sphere. The conditions (‡) ensure D is a simply connected domain on $\mathbb{D}$, and clearly $P(D) \cap \bar{C} = \phi$. Moreover neither the k-cycle nor ∞ belong to D, so D does not contain any accumulation point of successors of a point outside B_λ. Hence Cremer's theorem applies to $\{R_\omega^n(z)\}_{n=1}^\infty$ over D.

Let ε denote the exterior of a closed disk centered at the origin, containing A_k, and of a radius so large that $T_\lambda \varepsilon \subset \varepsilon$. Define a region S on $\mathbb{D}$ to consist of $\mathbb{C} - A_k - \varepsilon$ lifted to one of the spheres of $\mathbb{D}$, together with $(\mathbb{C} - A_k - \varepsilon) \cap \gamma$ lifted to the other sphere, such that $S \subset D$. Then Cremer's theorem applies to $\{R_\omega^n(z)\}_{n=1}^\infty$ over S.

Observe now that $R_\pm S \subset S$. This is true because $\mathbb{C} - A_k$ is totally invariant under T_λ, and $R\varepsilon \subset \varepsilon$. It follows that $R_\omega^{n+1}(D) \subset R_\omega^n(D)$, and so $\{R_\omega^n(D)\}_{n=1}^\infty$ converges uniformly to a single constant limit belonging to B_λ. The convergence is uniform because S is closed. The last part of the theorem follows from the last part of Cremer's theorem. ∎

Theorem 2.[2] Let γ be the negative axis branch cut, and $2 < \lambda < \infty$. Let $S = [\lambda - 1/2 - (\lambda + 1/4)^{1/2}, \lambda + 1/2 + (\lambda + 1/4)^{1/2}]$ and let D be any bounded open set in $\mathbb{C}$ such that $S \subset D$, $D \cap (-\infty, 0] = \phi$ and $D \cap [\lambda^2, \infty) = \phi$. Then for each $\tilde{\omega} \in \Omega$, $\{\tilde{R}^n_\omega(z)\}_{n=1}^{\infty}$ converges uniformly on D to a single element $\tilde{b}(\tilde{\omega}) \in B_\lambda$. The mapping $\tilde{b}: \Omega \to B_\lambda$ is one-to-one and onto.

Sketch of the Proof. See also Ref. 2. A similar argument to the proof of Theorem 1 applies here. In this case, the union of the branch cuts $\{T^n_\lambda \gamma \mid n \in \{0, 1, 2,...\}$ consists of the negative real axis together with the positive real axis from λ^2 to ∞. Hence D obeys the conditions of Cremer's theorem. Also, since $\tilde{R}_\pm S \subset S$ we can assume without loss of generality that $\tilde{R}_\pm D \subset D$, which gives the desired convergence of $\{\tilde{R}_{\tilde{\omega}}(z)\}_{n=1}^{\infty}$. The one-to-one property of the resulting mapping $\tilde{b}$ from Ω into $B\lambda$ follows from the fact that

$$S \cap \bigcup_{n=0}^{\infty} T^n_\lambda \gamma = \phi \quad \blacksquare$$

Theorem 3. Let γ be the positive axis branch cut and $-\infty < \lambda < -1/4$. Let S be any closed bounded simply connected domain. Then for each $\omega \in \Omega$, $\{R^n_\omega(z)\}_{n=1}^{\infty}$ converges uniformly on S to a single element $b(\omega) \in B_\lambda$. The mapping $b: \Omega \to B_\lambda$ is one-to-one and onto.

Sketch of the Proof. Without loss of generality we can take S to be a closed disk, centered at the origin of radius so large that $T_\lambda S \supset S$. Let D be an open disk which contains S, such that $T_\lambda D \supset D$. Then there is a finite integer m such that $R^m D \cap \{\bigcup_{n=0}^{\infty} T^n_\lambda \gamma\} = \phi$ [since $B_\lambda \cap \mathbb{R} = \phi$, because $T_\lambda(z) - z > 0$ for $z \in \mathbb{R}$, and B_λ is the closure of repulsive cycles] and $R^m D$ is simply connected. Letting $\tilde{D} = R^m D$, we find that Cremer's theorem applies to $\{R^n_\omega(z)\}_{n=1}^{\infty}$ over $\tilde{D}$. Moreover, since $R_\pm(R^m S) \subset R^m S$, it follows that $\{R^n_\omega(z)\}_{n=1}^{\infty}$ converges uniformly to a single constant in $b(\omega)$, for $z \in R^m S$ and consequently for $z \in S$. The one-to-one property follows from $R^m S \cap \{\bigcup_{n=0}^{\infty} T^n_\lambda \gamma\} = \phi$. This complete the sketch of the proof. $\blacksquare$

In order to make a specific definition of the λ-chains $b(\omega)$ or $\tilde{b}(\omega)$, under the conditions of Theorems 1, 2, and 3, we will suppose the value of z, required in Definition 3, to lie in the upper half-plane.

We will have occasion to use the notation

$$b(\omega) = \lambda + e_1\sqrt{(}\lambda + e_2\sqrt{(}\lambda + e_3 ...$$

when $\omega = (e_1, e_2, e_3,...) \in \Omega$, and the positive axis branch cut is understood. A similar notation may be used for negative axis λ-chains. Observe that λ-chains can be used not only to characterize elements of B_λ, but also as a means for numerical computation.

2.3. Relation Between Positive Axis and Negative Axis λ-Chains

When $\lambda < -1/4$, b is one-to-one; while when $\lambda > 2$, $\tilde{b}$ is one-to-one. As λ increases from $-1/4$ to 2 more and more elements of B_λ lie on the positive axis and it is necessary to convert these elements from a description in terms of positive axis λ-chains to one in terms of negative axis λ-chains, in order to preserve one-to-oneness. This is because there is more than one λ-chain characterization of an element of B_λ which lies on the branch-cut for the λ-chain; this leads in particular to computational instability, and in such cases we say that the λ-chain suffers from "branch-cut instability."

The conversion between positive axis and negative axis λ-chains is effected using a mapping $h: \Omega \to \Omega$ defined by

$$h(\omega) = h(e_1, e_2, e_3, \ldots) = (e_1 e_2, e_2 e_3, e_3 e_4, \ldots) = \tilde{\omega}$$

h is two-to-one and onto, and the two solutions of $h(\omega) = \tilde{\omega} = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \ldots)$ are $\pm\omega$ with $\omega = (e_1, e_2, e_3, \ldots)$ and $e_1 = 1$, $e_{j+1} = \tilde{e}_j e_j$ for $j \in \{1, 2, 3, \ldots\}$.

Definition 5. The λ-chains $b(\omega)$ and $\tilde{b}(\tilde{\omega})$ *correspond* when $\tilde{\omega} = h(\omega)$.

One would like to say that if $b(\omega)$ and $\tilde{b}(\tilde{\omega})$ correspond then they represent the same element of B_λ. The actual situation is more interesting, as shown by the following statement and its proof.

Theorem 4. Let $\tilde{\omega} = h(\omega)$. If $\tilde{\omega}$ has infinitely many -1's then

$$\tilde{b}(\tilde{\omega}) = \{b(\omega), b(-\omega)\}$$

If $\tilde{\omega}$ has a finite number n of -1's, and if $\omega = (1, e_2, e_3, \ldots)$, then

$$\tilde{b}(\tilde{\omega}) = b((-1)^n \omega)$$

Proof. Let $\omega \in \Omega$ be given, and recall that $b(\omega)$ consists of the set of accumulation points of $\{R^n(z)\}_{n=1}^{\infty}$. Thus, taking

$$\omega = (e_1, e_2, e_3, \ldots)$$

$b(\omega)$ equals the limits of the sequence

$$\lambda + e_1\sqrt{z}, \lambda + e_1\sqrt{(\lambda + e_2\sqrt{z})}, \lambda + e_1\sqrt{(\lambda + e_2\sqrt{(\lambda + e_3\sqrt{z})})}, \ldots$$

(positive axis cut)

where the positive axis branch cut is understood. Recalling that z has been fixed to lie in the upper half-plane, it is straightforward to verify that the latter is the same as the sequence of number $\{\tilde{S}^n_\omega(z)\}_{n=1}^\infty$ whose successive terms are

$$\lambda + e_1\sqrt{z}, \lambda + e_1 e_2\sqrt{(\lambda + e_2\sqrt{z})}, \lambda + e_1 e_2\sqrt{(\lambda + e_2 e_3\sqrt{(\lambda + e_3\sqrt{z})})}, \ldots$$
$$\text{(negative axis cut)}$$

where the negative axis branch cut is understood. Compare this with the sequence

$$\lambda + e_1 e_2\sqrt{z}, \lambda + e_1 e_2\sqrt{(\lambda + e_2 e_3\sqrt{z})}, \lambda + e_1 e_2\sqrt{(\lambda + e_2 e_3\sqrt{(\lambda + e_3 e_4\sqrt{z})})}, \ldots$$

whose limit points form the set $\tilde{b}(\tilde{\omega})$. For each n we have either $e_n e_{n+1} = e_n$ or $e_n e_{n+1} = -e_n$; accordingly, for each n, we have $R^n_{\tilde{\omega}}(z) \in \{\tilde{S}^n_\omega(z), S^n_{-\omega}(z)\}$. Hence if $\tilde{\omega}$ does not contain only finitely many -1's, which means ω has infinitely many $+1$'s and infinitely many -1's, it follows that

$$\tilde{b}(\tilde{\omega}) = \{b(\omega), b(-\omega)\}$$

If $\tilde{\omega}$ contains only finitely many -1's; then either ω contains only finitely many -1's, in which case

$$\tilde{b}(\tilde{\omega}) = b(\omega)$$

or else ω contains only finitely many $+1$'s, in which case

$$\tilde{b}(\tilde{\omega}) = -b(\omega)$$

The latter sentence is equivalent to the last assertion in the theorem. ∎

We remark that $b(-\omega)$ is the complex conjugate of $b(\omega)$, see Ref. 2.

The following statements can now be established with the help of Theorems 1, 2, 3, and 4, and the properties of $h: \Omega \to \Omega$. They are valid when either $-\infty < \lambda < -1/4$, or $2 < \lambda < \infty$, or $-1/4 < \lambda < 2$ and T_λ has an attractive k-cycle. Both $b: \Omega \to B_\lambda$ and $\tilde{b}: \Omega \to B_\lambda$ are onto, and b is single-valued. If $z = \tilde{b}(\omega)$ for some $\omega \in \Omega$, then $\bar{z} \in \tilde{b}(\omega)$. If $b(\omega) \in \mathbb{R}$ for some $\omega \in \Omega$ then there is $\sigma \in \Omega$ with $\sigma \neq \omega$ such that $b(\omega) = b(\sigma)$. When $-\infty < \lambda < -1/4$, $\tilde{b}$ is double-valued and b is one-to-one. When $\lambda > 2$, $\tilde{b}$ is single-valued and one-to-one, while b is two-to-one. As λ increases from less than $-1/4$ to greater than 2, $\tilde{b}$ changes from double-valued to single-valued, while b changes from one-to-one to two-to-one. These changes mark the progression of B_λ from having the property $B_\lambda \cap \mathbb{R} = \phi$ when $-\infty < \lambda < -1/4$ to having the property $B_\lambda \cap \mathbb{R} = B_\lambda$ when $2 < \lambda < \infty$.

2.4. Equivalence Classes of λ-Chains and the Topology of B_λ

Define the distance between $\omega = (e_1, e_2, e_3, \ldots)$ and $\sigma = (f_1, f_2, f_3 \ldots)$ in Ω by $|\omega - \sigma| = |\sum_{i=1}^{\infty} (e_i - f_i)/2^{i+1}|$. Then Ω is a topological space homeomorphic to the real interval $[0, 1]$, provided that we identify the elements $(e_1, e_2, \ldots, e_m, 1, -1, -1, \ldots)$ and $(e_1, e_2, \ldots, e_m, -1, +1, +1, \ldots)$, whose distance apart is zero.

Theorem 5. When $-1/4 < \lambda < 2$ and T_λ has an attractive k-cycle, b: $\Omega \to B_\lambda$ is continuous.

Proof. First we show that b is well-defined with respect to the identifications in Ω. Observe that $b(+1, +1, +1, \ldots) = b(-1, -1, -1, \ldots) \in \gamma$; see Ref. 2. Hence $b(+1, -1, -1, -1, \ldots) = b(-1, +1, +1, +1, \ldots)$, which lies on the negative real axis. All preimages of the latter point do not lie on γ, whence $b(e_1, e_2, \ldots, e_m, 1, -1, -1, -1, \ldots) = b(e_1, e_2, \ldots, e_m, -1, +1, +1, +1, \ldots)$ and so b is well defined.

Let $\omega \in \Omega$ and $\varepsilon > 0$. Introduce the projection operator P_m: $\Omega \to \Omega$, defined by $P_m(e_1, e_2, e_3, \ldots) = (e_1, e_2, e_3, \ldots, e_m, -1, -1, -1, \ldots)$. By Theorem 1 there is an integer N such that $|b(\omega) - R_\omega^n(z)| < \varepsilon$ for all $n \geqslant N$ and $z \in B_\lambda$. Hence $|b(\omega) - b(\sigma)| < \varepsilon$ whenever $P_N\omega = P_N\sigma$, with $\sigma \in \Omega$.

Suppose ω does not terminate in $(+1, +1, +1, \ldots)$ or $(-1, -1, -1, \ldots)$. Then we can choose $\delta > 0$ so that $|\omega - \sigma| < \delta$ implies $P_N\omega = P_N\sigma$, and hence that $|b(\omega) - b(\sigma)| < \varepsilon$.

Suppose ω does terminate in $(+1, +1, +1, \ldots)$ or $(-1, -1, -1, \ldots)$. Then ω possesses two equivalent representations ω and ω', one terminating $(+1, +1, +1, \ldots)$ and the other terminating $(-1, -1, -1, \ldots)$. Note that $b(\omega) = b(\omega')$. Choose the positive integer M so that $|b(\omega') - R_{\omega'}^m(z)| < \varepsilon$ whenever $z \in B_\lambda$ and $m \geqslant M$. It follows that $|b(\omega) - b(\sigma)| < \varepsilon$ whenever $P_M\omega' = P_M\sigma$. Finally observe that we can pick $\delta > 0$ such that $|\omega - \sigma| < \delta$ implies either $P_N\omega = P_N\sigma$ or $P_M\omega' = P_M\sigma$, in both of which cases $|b(\omega) - b(\sigma)| < \varepsilon$. ∎

In what follows we assume b: $\Omega \to B_\lambda$ is continuous. We then have a useful description of the topology of B_λ in terms of positive axis λ-chains. b: $\Omega \to B_\lambda$ is a continuous mapping of a compact topological space onto a Hausdorff space. Hence the identification topology of B_λ which is induced by b is the same as the relative topology of B_λ as a subset of $\mathbb{C}$.[33] That is, for any subset $O \subset B_\lambda$ we have that $b^{-1}O$ is open if and only if there is an open subset $Q \subset \mathbb{C}$ such that $O = Q \cap \mathbb{C}$.

Let us consider the construction of some continuous curves lying in B_λ, which join a given pair of points z_1 and z_2. It will be convenient for us to identify each element $\omega = (e_1, e_2, e_3, \ldots)$ of Ω with the corresponding element $.\theta(e_1)\,\theta(e_2)\,\theta(e_3)\ldots$ of $[0, 1]$ in binary decimal expansion, where $\theta(+1) = 1$ and $\theta(-1) = 0$. Then we refer to $[0, 1]$ in place of Ω. Also, when $\delta < \gamma$, we will understand by $[\gamma, \delta]$ the usual closed interval $[\delta, \gamma]$. Let $\alpha \in b^{-1}(z_1)$, $\beta \in b^{-1}(z_2)$, and form $P = [\alpha, \beta_1] \cup [\alpha_2, \beta_2] \cup \cdots \cup [\alpha_{n-1}, \beta_{n-1}] \cup [\alpha_n, \beta]$, where $\alpha_{i+1} \in b^{-1}(b(\beta_i))$ and $\beta_i \in [0, 1]$ for $i \in \{0, 1, 2, \ldots, n-1\}$. Then $b(P)$ is a continuous path which lies in B_λ and joins z_1 to z_2. For example, we know that $b(0) = b(1)$, and hence each of $P_1 = [1/3, 0] \cup [1, 2/3]$ and $P_2 = [1/3, 2/3]$ leads to a continuous path which lies in B_λ and joins $b(1/3)$ to $b(2/3)$.

If Γ is a continuous curve in B_λ which connects z_1 to z_2, then its complement $B_\lambda - \Gamma$ is open and $b^{-1}(B_\lambda - \Gamma) = [0, 1] - b^{-1}(\Gamma)$ must consist of a countable union of open intervals in $[0, 1]$.

Definition 6. Elements σ and ω of Ω are *equivalent* when $\omega \in b^{-1}(b(\sigma))$. Equivalent elements are denoted $\sigma \sim \omega$.

When $b\colon \Omega \to B_\lambda$ is continuous we can think of the topology of B as being that of $[0, 1]$ "pinched together" or "joined to itself" at equivalent points. For example, when $-1/4 < \lambda < 3/4$ one can show that the only pair of distinct points in $[0, 1]$ which are equivalent is $\{0, 1\}$, and as a consequence B_λ is a simple Jordan curve. In Section 3.1 we describe the dependence on λ of the equivalence classes of points in Ω.

2.5. Symbolic Dynamics Using λ-Chains

Not only do the λ-chains codify the topology of B_λ, but also they describe the dynamics of $T_\lambda\colon B_\lambda \to B_\lambda$. Let $\mathscr{E}\colon \Omega \to \Omega$ denote the right-shift operator defined by

$$\mathscr{E}(e_1, e_2, \ldots, e_n, \ldots) = (e_2, e_3, \ldots, e_{n-1}, \ldots)$$

When either $-\infty < \lambda < -1/4$, or $2 < \lambda < \infty$, or $-1/4 < \lambda < 2$ and T_λ has an attractive k-cycle, we have

$$T_\lambda b(\omega) = b(\mathscr{E}\omega) \qquad \text{for all } \omega \in \Omega$$

Similarly, for the negative axis λ-chains $\tilde{b}(\omega)$, which are single-valued when $\tilde{b}(\omega)$ is real and double-valued otherwise, we have

$$\{T_\lambda \tilde{b}(\omega)\} = \{\tilde{b}(\mathscr{E}\omega)\} \qquad \text{for all } \omega \in \Omega$$

where the brackets $\{\ \}$ denote the set of values of the enclosed set-valued function. These relations are readily proved. For example, when Theorem 1 applies, since $T_\lambda\colon \mathbb{C} \to \mathbb{C}$ is continuous,

$$\begin{aligned} T_\lambda b(\omega) &= T_\lambda \lim_{n\to\infty} R^n_\omega(z) = \lim_{n\to\infty} T_\lambda R^n_\omega(z) \\ &= \lim_{n\to\infty} R^n_{\mathscr{E}\omega}(z) = b(\mathscr{E}\omega) \end{aligned}$$

where $z \in B_\lambda$. The corresponding result for $\tilde{b}$ follows from

$$\{T_\lambda \tilde{b}(\omega)\} = T_\lambda b(\{h^{-1}(\omega)\}) = b(\mathscr{T}\{h^{-1}(\omega)\})$$
$$= b(\{h^{-1}(\mathscr{T}\omega)\}) = \{\tilde{b}(\mathscr{T}\omega)\}$$

Notation. Let us denote periodic elements of Ω according to

$$(f_1, f_2,..., f_l, f_1, f_2,..., f_l, f_1,...) = (f_1, f_2,..., f_l)$$

and eventually periodic elements of Ω according to

$$(e_1, e_2,..., e_k, f_1, f_2,..., f_l, f_1, f_2,..., f_l, f_1,...)$$
$$(e_1, e_2,..., e_k \} f_1, f_2,..., f_l).$$

Also, it is convenient to conserve only the signs of components of elements of Ω. Thus, for example, we write

$$(+1, -1, +1, -1, +1, -1,...) = (+-)$$

and

$$(-1, +1, +1, -1, +1, -1, +1, -1,...) = (-+\} +-)$$

It is now possible to describe dynamical features of T_λ acting on B_λ in terms of λ-chains. To illustrate this, take $\lambda < -1/4$ so that b: $\Omega \to B_\lambda$ is one-to-one. Then the only 2-cycle of T_λ on B_λ must be $\{z_1, z_2\}$, where $a_1 = b(+-)$ and $z_2 = b(-+)$. Any point $z \in B_\lambda$ such that $T_\lambda^n z = z_1$ must be expressible in the form $z = b(e_1, e_2,..., e_k \} +-)$. The only 3-cycles of T_λ on B_λ must be $\{b(++-), b(+-+), b(-++)\}$, and $\{b(--+), b(-+-), b(+--)\}$. The only fixed point (1-cycles) of T_λ on B_λ are $b(+)$ and $b(-)$. Observe that there are exactly 2^n distinct elements $\omega \in \Omega$ such that $\mathscr{T}^n\omega = \omega$. Hence, when either $b: \Omega \to B_\lambda$ or $\tilde{b}: \Omega \to B_\lambda$ is one-to-one, B_λ contains exactly 2^n distinct points z such that $T_\lambda^n(z) = z$, and since the polynomial $T_\lambda^n(z) - z = 0$ possesses at most 2^n distinct roots, we conclude that all k-cycles for all k belong to B_λ. Conversely, when T_λ possesses an attractive k-cycle neither $b: \Omega \to B_\lambda$ not $\tilde{b}: \Omega \to B_\lambda$ is one-to-one.

When $-1/4 < \lambda < 2$ and T_λ possesses an attractive k-cycle, the representation of cycles belonging to B_λ is more complicated. We have $T_\lambda^n b(\omega) = b(\omega)$ if and only if $b(\mathscr{T}^n\omega) = b(\omega)$, if and only if $\omega \sim \mathscr{T}^n\omega$. Hence there is an interplay between the equivalence class structure of Ω (which, we recall, fixes the topology of B_λ) and the dynamics of T_λ on B_λ.

3. RELATION BETWEEN THE ITERATED REAL MAP $X \to (X - \lambda)^2$ AND λ-CHAINS

3.1. How the Equivalence Classes of λ-Chains are Fixed by $B_\lambda \cap [0, \infty)$

Throughout this section we suppose that $-1/4 < \lambda < 2$ and that T_λ possesses an attractive k-cycle. Under these conditions we know from Section 2.4 that the topology of B_λ is that of $[0, 1]$ "pinched together" or "joined to itself" at equivalent points. Here we show how the set of equivalent points are determined by the elements of $B_\lambda \cap [0, \infty)$.

For $m \in \{1, 2, 3,...\}$ let P_m: $\Omega \to \Omega$ denote the projection operator such that

$$P_m(e_1, e_2, e_3,...) = (e_1, e_2,..., e_m \wr -)$$

and let $P_0\omega = (-)$. γ denotes the positive axis branch cut, $\gamma = [0, \infty)$.

Theorem 6. Let $-1/4 < \lambda < 2$, and let T_λ have an attractive k-cycle. Let $z \in B_\lambda$. If $T_\lambda^n(z) \notin \gamma$ for all $n \in \{0, 1, 2,...\}$ then $\{b^{-1}(z)\}$ consists of a single element. If $T_\lambda^n(z) \in \gamma$ but $T_\lambda^{n-1}(z) \notin \gamma$ for some $n \in \{1, 2, 3,...\}$, then $\{b^{-1}(z)\} = \{\omega_1, \omega_2\}$ where $\omega_1 \neq \omega_2$, $P_{n-1}\omega_1 = P_{n-1}\omega_2$, and $\{\mathscr{S}^{n-1}\omega_1, \mathscr{S}^{n-1}\omega_2\} = \{h^{-1}(h(\mathscr{S}^{n-1}\omega_1))\}$. If $z \in \gamma$ then $\{b^{-1}(z)\} = \{\omega_1, \omega_2\} = \{h^{-1}(h(\omega_1))\}$.

Example 1. Let $-1/4 < \lambda < 3/4$. Then $B_\lambda \cap [0, \infty)$ contains only $a = \lambda + 1/2 + (\lambda + 1/4)^{1/2}$, and $b^{-1}(a) = \{(+), (-)\}$. Theorem 6 now states that the only elements in Ω whose equivalence classes consist of more than one element are $(+) \sim (-)$, $(+ \wr -) \sim (- \wr +)$, and $(e_1, e_1,..., e_n, + \wr -) \sim (e_1, e_2,..., e_n, - \wr +)$. It was precisely these equivalences which permitted the identification of $[0, 1]$ with Ω in Theorem 5.

Proof of Theorem 6. Suppose $T_\lambda^n(z) \notin \gamma$ for all n. Then $T_\lambda^n(z) \notin \mathbb{R}$ for all n, since if $T_\lambda^m z < 0$ for some m then $T_\lambda^{m+1}(z) \in \gamma$, and $T_\lambda^n(z) \neq 0$ because the branch point 0 is attracted to the k-cycle and so does not belong to B_λ. It follows that, for each n, either Im $T_\lambda^n z > 0$ or Im $T_\lambda^n z < 0$. The coefficients in $\omega = (e_1, e_2,...) \in b^{-1}(z)$ are given by $e_n = +1$ if Im $T_\lambda^n(z) > 0$ and $e_n = -1$ if Im $T_\lambda^n(z) < 0$, which fixes ω uniquely.

Suppose $T_\lambda^n(z) \in \gamma$ but $T_\lambda^{n-1}(z) \notin \gamma$ for some $n \in \{1, 2, 3,...\}$. Then $T_\lambda^{n-1}(z) < 0$, and for each $k \in \{0, 1,..., n-2\}$ either Im $T_\lambda^n(z) > 0$ or Im $T_\lambda^k(z) < 0$, which fixes uniquely the coefficients in $P_{n-1}\omega$ independently of the choice of $\omega \in \{b^{-1}(z)\}$. Since $T_\lambda^m(z)$ for $m \geqslant n-1$ lies on the real line, it is convenient to consider the associated negative axis λ-chains. Let $\tilde{\omega} = (\tilde{e}_1, \tilde{e}_2,...) \in \{\tilde{b}^{-1}(T_\lambda^{n-1}z)\}$. Then $T_\lambda^{n-1}z < 0$ implies $\tilde{e}_1 = -1$. Moreover $\tilde{e}_j$ for $j \in \{1, 2, 3,...\}$ is uniquely defined by $\tilde{e}_j = +1$ when $T_\lambda^{n+j-2}(z) > \lambda$ and $\tilde{e}_j = -1$ when $T_\lambda^{n+j-2}z < \lambda$. (Notice that $T_\lambda^n z \neq \lambda$ for any n because $0 \notin B_\lambda$

implies $\lambda = T_\lambda 0 \notin B_\lambda$.) Hence $\tilde{\omega}$ is fixed uniquely, and $\{b^{-1}(T_\lambda^{n-1}z)\} = \{h^{-1}\tilde{\omega}\} = \{h^{-1}(h\mathscr{S}^{n-1}\omega)\}$ consists of exactly two elements, as claimed.

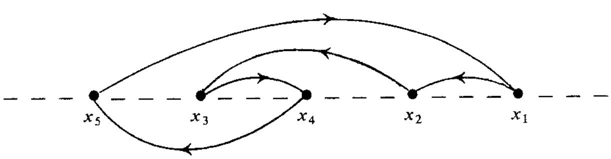
Similarly, if $z \in \gamma$ then $\tilde{b}^{-1}(z)$ has only one element and $\{b^{-1}(z)\} = \{h^{-1}(\tilde{b}^{-1}(z))\}$ consists of two elements $\{\omega_1, \omega_2\} = \{h^{-1}(h(\omega_1))\}$. ∎

3.2. Orders of Visitation for Real λ-Chains: Loners and Partners

In view of Theorem 6 we want to describe $B_\lambda \cap [0, \infty)$. To do this we need to know what is meant by order of visitation for a real k-cycle given by a λ-chain, and how to calculate it. Because we are concerned with elements of the positive axis branch cut, it is most convenient to work in terms of negative axis λ-chains.

Let $\omega = (e_1, e_2, \ldots, e_k)$ be a periodic element of Ω which may contain subperiods, and let λ be such that $\tilde{b}(\omega)$ is defined and real. Then by the cycle $(\tilde{b}(e_1, e_2, \ldots, e_k))$ we mean the set of points $\{\tilde{b}(\omega), \tilde{b}(\mathscr{S}\omega), \ldots, \tilde{b}(\mathscr{S}^{k-1}\omega)\}$. In general, when we refer to such a cycle it is to be understood that λ is such that the cycle is real. If ω belongs to a k-cycle in $\Omega(\mathscr{S}^j\omega \neq \omega$ for $j \in \{1, 2, \ldots, k-1\})$ then $(\tilde{b}(e_1, e_2, \ldots, e_k))$ is a real k-cycle for T_λ. We denote this k-cycle by $\{x_1, x_2, \ldots, x_k\}$ where $x_i = \tilde{b}(\mathscr{S}^{i-1}\omega)$, so that $T_\lambda x_j = x_{j+1}$ for $j \in \{1, 2, \ldots, k-1\}$ and $T_\lambda x_k = x_1$. Since the k-cycle $(\tilde{b}(e_1', e_2', \ldots, e_k'))$ is the same as the k-cycle $(\tilde{b}(e_1, e_2, \ldots, e_k))$ whenever $(e_1', e_2', \ldots, e_k')$ is a cyclic permutation of $(e_1, e_2, \ldots, e_k)$, we can assume without loss of generality that $x_1 < x_j$ for $j \in \{2, 3, \ldots, k\}$.

There are two logical orderings of a real k-cycle. Already we have used the notation $\{x_1, x_2, \ldots, x_k\}$ putting the points in iterative order. The points also may be put in increasing order. For example a 5-cycle may have the increasing order $x_5 < x_3 < x_4 < x_2 < x_1$. We call the combined information the order of visitation. It can be given diagrammatically; for example



To determine the order of visitation of a k-cycle $(\tilde{b}(e_1, e_2, \ldots, e_k))$ we define a mapping ϕ from $\sigma = (f_1, f_2, f_3, \ldots) \in \Omega$ into $[0, 1]$ in binary decimal representation by

$$\phi(\sigma) = 0 \cdot \alpha_1 \alpha_2 \alpha_3 \cdots \alpha_k \cdots$$

where

$$\alpha_i = \begin{cases} 1 & \text{if } f_1 f_2 \cdots f_i = +1 \\ 0 & \text{if } f_1 f_2 \cdots f_i = -1 \end{cases}$$

Theorem 7. (A) The order of visitation of the real k-cycle $(b(\omega)) = \{x_1, x_2,..., x_k\}$ is given by the increasing order of the set of real numbers $\{\phi(\omega), \phi(\mathscr{E}\omega),..., \phi(\mathscr{E}^{k-1}\omega)\}$. If $\omega = (e_1, e_2,..., e_k)$ and $x_1 < x_j$ for $j \in \{2, 3,..., k\}$, then $e_i = +1$ when $x_i > x_k$ and $e_i = -1$ when $x_i < x_k$. (B) Conversely, if $(\tilde{e}_1, \tilde{e}_2,..., \tilde{e}_k)$ is a k-cycle in Ω such that $\tilde{e}_i = +1$ when $x_i > x_k$ and $\tilde{e}_i = -1$ when $x_i < x_k$, then the real k-cycle $(\tilde{b}(\tilde{e}_1, \tilde{e}_2,..., \tilde{e}_k))$ has the same order of visitation as $\{x_1, x_2,..., x_k\}$.

This theorem allows us to make the following definition.

Definition 7. In the notation of Theorem 7, when $\tilde{e}_k = -e_k$ and $(\tilde{b}(\tilde{e}_1, \tilde{e}_2,..., \tilde{e}_k)) = \{\tilde{x}_1, \tilde{x}_2,..., \tilde{x}_k\}$ is a real k-cycle, the two real k-cycles $\{x_1, x_2,..., x_k\}$ and $\{\tilde{x}_1, \tilde{x}_2,..., \tilde{x}_k\}$ are partners. Otherwise $\{x_1, x_2,..., x_k\}$ is a *loner.*

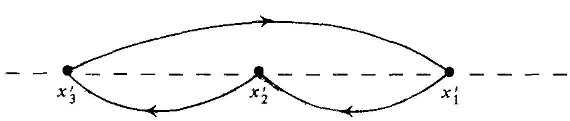
The point of the definition is this. At fixed λ the theorem tells us that when $\{x_1, x_2,..., x_k\}$ is a loner there is no other real k-cycle, given by a negative axis λ-chain, with the same order of visitation; and when $\{x_1, x_2,..., x_k\}$ has a partner, then its partner is the only other real k-cycle, given by a negative axis λ-chain, with the same order of visitation. The significance of a k-cycle being a loner, or having a partner, is described in Section 3.3.

Proof of Theorem 7(A). It is straightforward to check that if $\tilde{b}(\sigma_1)$ and $\tilde{b}(\sigma_2)$ are real λ-chains then $\tilde{b}(\sigma_1) < \tilde{b}(\sigma_2)$ if and only if $\phi(\sigma_1) < \phi(\sigma_2)$. From this follows the first statement in the theorem.

Since x_1 is the smallest member of the k-cycle, it is closest to zero. Hence x_k is closest to λ. Hence, x_j with $j \neq k$ is greater than λ if and only if x_j is greater than x_k, and x_j is less than λ if and only if x_j is less than x_k. But $x_j > \lambda$ if and only if $e_j = +1$, and $x_j < \lambda$ if and only if $e_j = -1$, which proves the second statement in the theorem.

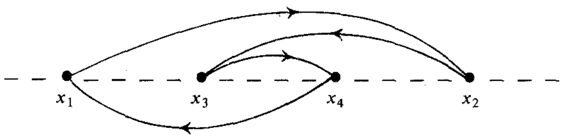
We defer the proof of (B) until after the proof of Theorem 9. Although a direct combinatorial proof should be available, the only approach we know relies on the structure of the bifurcation diagram for T_λ.

Example 2. Consider the real 3-cycle $(\tilde{b}(++-)) = \{x'_1 = \tilde{b}(++-), x'_2 = \tilde{b}(+-+), x'_3 = \tilde{b}(-++)\}$. The order of visitation is given by the ordering of the sequence $\{.110, .100, .000\}$, whence $x'_3 < x'_2 < x'_1$, which can also be expressed

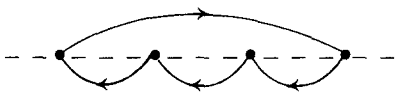


If we relabel the cycle $x_1 = x_3'$, $x_2 = x_1'$, $x_3 = x_2'$, then the order of visitation is $x_1 < x_3 < x_2$, where $x_1 = \tilde{b}(-++)$. Theorem 6 (B) now asserts that the order of visitation for the 3-cycle $(\tilde{b}(-+-)) = \{\tilde{x}_1 = \tilde{b}(-++), \tilde{x}_2 = \tilde{b}(++-), x_3 = \tilde{b}(+-+)\}$ is $\tilde{x}_1 < \tilde{x}_3 < \tilde{x}_2$, which is readily checked. The two cycles $\{x_1, x_2, x_3\}$ and $\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$ is this example are *partners*.

Example 3. Consider the real 4-cycle $(\tilde{b}(-+--)) = \{x_1 = \tilde{b}(-+--), x_2 = \tilde{b}(+---), x_3 = \tilde{b}(---+), x_4 = \tilde{b}(--+-)\}$. The order of visitation is given by the ordering of the sequence of numbers $\{.0010, .1010, .0100, .0110\}$, whence $x_1 < x_3 < x_4 < x_2$, which can also be expressed



In this case $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_3) = (-1, +1, -1, +1)$ which is not a 4-cycle in Ω. Hence $\{x_1, x_2, x_3, x_4\}$ is a loner. On the other hand, the two real 4-cycles $(\tilde{b}(-++-))$ and $(\tilde{b}(-+++))$ are partners, each with the order of visitation



3.3. Continuation of Real k-Cycles

Here we consider the analytic continuation, especially through decreasing λ-values, of negative axis λ-chains for real k-cycles, starting from a value of λ greater than 2. We show how combinatorial features of the λ-chain allow one to compute its complete history, including whether or not it was an attractive cycle for some range of λ-values, whether it first became real via tangent or pitch-fork bifurcation, and the λ-chains for cycles from which it bifurcated. When a k-cycle is attractive it cannot be represented by a λ-chain but instead obeys a functional equation which is determined from the corresponding λ-chain when $\lambda > 2$. Of particular interest to us is the switching between elements of Ω which occurs when a repulsive k-cycle leaves the Julia set to become an attractive cycle (what elements of the Julia set does the old λ-chain now represent?), and vice versa.

The actual continuation arguments we use are of standard type (see for example Refs. 23, 35), and we rely on the works of Guckenheimer (25) and Douady and Hubbard (20) to tell us how cycles are born, and that a real cycle cannot become not real for some larger λ-value.

Let $\omega = (e_1, e_2, \ldots, e_k)$ be a k-cycle in Ω, and fix $\lambda_0 > 2$. For this value of λ, $(\tilde{b}(e_1, e_2, \ldots, e_k))$ is a real k-cycle because $B_{\lambda_0} \subset \mathbb{R}$. We denote this k-cycle by $\{x_1^0, x_2^0,..., x_k^0\}$, where $x_1^0 = \tilde{b}(e_1, e_2,..., e_k)$ and we suppose $x_1^0 < x_j^0$ for $j \in \{2, 3,..., k\}$. We define a function of x and λ by $(e_1, e_2,..., e_k)(x, \lambda) := \lambda + e_1\sqrt{(\lambda + e_2\sqrt{(\lambda + \cdots e_{k-1}\sqrt{(\lambda + e_k\sqrt{x})}\cdots)})}$ where the negative axis branch cut is used, and where the domain is the set of real λ and x such that $(e_1, e_2,..., e_k)$ (x, λ) is real. Then (x_j^0, λ_0) is a solution of the equation $u_j(x, \lambda) = 0$, for $j \in \{1, 2,..., k\}$, where

$$u_j(x, \lambda) := x - (e_j, e_{j+1},..., e_k, e_1, e_2,..., e_{j-1})(x, \lambda)$$

The implicit function theorem states that there is a unique continuation of this solution when $\partial u_j(x_j, \lambda)/\partial x_j$ exists and does not vanish.

We have

$$\frac{\partial u_j}{\partial x_j}(x_j, \lambda) = 1 - \frac{e_1 e_2 \cdots e_k}{2^k (x_1 x_2 \cdots x_k)^{1/2}} \tag{*}$$

It follows that there is an open interval I containing λ_0, and a unique set of continuous functions $\{x_1(\lambda), x_2(\lambda),..., x_k(\lambda)\}$ defined for $\lambda \in I$ and obeying $u_j(x_j(\lambda), \lambda) = 0$ for all $\lambda \in I$. Moreover (i) each of $\{x_1(\lambda), x_2(\lambda),..., x_k(\lambda)\}$ is finite and strictly positive for all $\lambda \in I$, and (ii)

$$\left[1 - \frac{e_1 e_2 \cdots e_k}{2^k (x_1 x_2 \cdots x_k)^{1/2}}\right] \neq 0$$

for all $\lambda \in I$. We assume I is the largest open interval such that the above statements are true.

We show that no two members of $\{x_1(\lambda), x_2(\lambda),..., x_k(\lambda)\}$ can be equal for $\lambda \in I$. We can have $x_j(\lambda) = x_l(\lambda)$ with $j \neq l$ only if the polynomial equation $T_\lambda^k(x) - x = 0$ has x_j as a double root. For this it is necessary that $\partial[T_\lambda^k(x_j) - x_j]/\partial x_j = 0$. That is,

$$2^k T'_\lambda(x_1)\, T'_\lambda(x_2) \cdots T'_\lambda(x_k) - 1 = 0$$

Since $u_j(x_j, \lambda) = 0$ and $x_j > 0$ for $j \in \{1, 2,..., k\}$, the last condition can be rewritten

$$e_1 e_2 \cdots e_k 2^k (x_1 x_2 \cdots x_k)^{1/2} - 1 = 0$$

which is not possible by (ii). We conclude $x_j(\lambda) \neq x_l(\lambda)$ for all $\lambda \in I$. It now follows directly from the equation $u_j(x_j(\lambda), \lambda) = 0$ that $\{x_1(\lambda), x_2(\lambda),..., x_k(\lambda)\}$ is a k-cycle for T_λ, with $x_1(\lambda) < x_j(\lambda)$ for $j \neq 1$ and having the same order of visitation as $\{x_1^0, x_2^0,..., x_k^0\}$.

It is important to understand that although $\{x_1^0, x_2^0, \ldots, x_k^0\} \subset B_{\lambda_0}$, it may not be true that $\{x_1(\lambda), x_2(\lambda), \ldots, x_k(\lambda)\} \subset B_\lambda$ when $\lambda < 2$. Observe that

$$\frac{\partial}{\partial x} T_\lambda^k(x)\Bigg|_{x=x_1(\lambda)} = e_1 e_1 \cdots e_k 2^k [x_1(\lambda)\, x_2(\lambda) \cdots x_k(\lambda)]^{1/2}$$

Hence $\{x_1(\lambda), x_2(\lambda), \ldots, x_k(\lambda)\}$ is a repulsive k-cycle and belongs to B_λ when $2^k[x_1(\lambda)\, x_2(\lambda) \cdots x_k(\lambda)]^{1/2} > 1$. $\{x_1(\lambda), x_2(\lambda), \ldots, x_k(\lambda)\}$ is a attractive k-cycle and does not belong to B_λ when $2^k[x_1(\lambda)\, x_1(\lambda) \cdots x_k(\lambda)]^{1/2} < 1$.

Let $I = (\nu, \mu)$. Then $\mu = \infty$. To see this we suppose that μ is finite and consider what can happen to $\{x_1(\lambda), x_2(\lambda), \ldots, x_k(\lambda)\}$ as $\lambda \to \mu$. No member of the cycle can approach ∞ as $\lambda \to \mu$, since any continuation of any finite root of $T_\lambda^k(x) - x = 0$ remains finite for finite λ. It follows that we can define $x_j(\mu) = \lim_{\lambda \to \mu} x_j(\lambda)$. We cannot have $x_1(\mu) = 0$ because all k-cycles belong to B_λ and $0 \notin B_\lambda$ for $\lambda > 2$. Finally, we cannot have

$$1 - \frac{e_1 e_2 \cdots e_k}{2^k [x_1(\mu)\, x_2(\mu) \cdots x_k(\mu)]^{1/2}} = 0$$

for if so then $T_\mu^k(x) - x = 0$ has a double root. The latter is not possible because it implies the false assertion that $\tilde{b}\colon \Omega \to B_\mu$ is not one-to-one. It now follows that $\{x_1(\mu), \ldots, x_k(\mu)\}$ has a unique continuation throughout some neighborhood of μ, where it obeys $u_j(x_j(\lambda), \lambda) = 0$. This contradicts our assumption on I.

We must have $\nu \geqslant -1/4$ because T_λ possesses no real k-cycles when $\lambda < -1/4$; also as above, we can define $x_j(\nu) = \lim_{\lambda \to \nu+} x_j(\lambda) < \infty$. Thus, on the continuation of solutions of $u_j(x_j(\lambda), \lambda) = 0$ we have so far established the following theorem.

Theorem 8. Let $\omega = (e_1, e_2, \ldots, e_k)$ be a k-cycle in Ω, let $\lambda = \lambda_0 > 2$, and let $(\tilde{b}(e_1, e_2, \ldots, e_k)) = \{x_1^0, x_2^0, \ldots, x_k^0\}$ be the corresponding real k-cycle in B_{λ_0}. Then there is a unique real continuation $(x_j(\lambda), \lambda)$ for λ in a largest open interval $I = (\nu, \infty)$ containing λ_0, where $\nu \geqslant -1/4$, such that

$$\begin{aligned} u_j(x_j(\lambda), \lambda) &= x_j(\lambda) \\ &\quad - (\lambda + e_j\sqrt{(\lambda + \cdots + e_k\sqrt{(\lambda + e_1\sqrt{(\lambda + \cdots + e_{j-1}\sqrt{x_j(\lambda)})\cdots))\cdots)}}} \\ &= 0 \end{aligned}$$

and $x_j(\lambda_0) = x_j^0$, for $j \in \{1, 2, \ldots, k\}$. $\{x_1(\lambda), x_2(\lambda), \ldots, x_k(\lambda)\}$ is a real k-cycle of T_λ, for all $\lambda \in I$.

When we can do so without ambiguity, we will use the notation $(\tilde{b}(e_1, e_2, \ldots, e_k))$ both for the real k-cycle $\{x_1(\lambda), x_2(\lambda), \ldots, x_k(\lambda)\}$ in

Theorem 8, and for its continuation through decreasing λ-values as long as it remains a real k-cycle, obeying $T_\lambda^k(x_1(\lambda)) = x_1(\lambda)$, even when it may be that the cycle in question is attractive (thus failing to belong to the Julia set) over a range of λ-values, and the negative axis λ-chain does not converge to the cycle with which it is identified by the notation. Notice that the order of visitation for the k-cycle is independent of λ and hence can be calculated from $\omega = (e_1, e_2, ..., e_k)$ as in Theorem 7A.

Before we consider what happens at ν, we recall some facts about the bifurcations of real solutions of $T_\lambda^k(x) - x = 0$. Fundamental is the Mandelbrot domain M, which consists of the set of points $\lambda \in \mathbb{C}$ such that B_λ is connected. The following information is drawn from the study of M by Douady and Hubbard.[20] (1) Let $\{x_1(\delta), x_2(\delta), ..., x_k(\delta)\}$ be a real k-cycle at $\lambda = \delta$. Then there is a real k-cycle $\{x_1(\lambda), x_2(\lambda), ..., x_k(\lambda)\}$ for all $\lambda \geqslant \delta$, depending continuously on λ.[40] (2) Let C denote a component of the interior $\mathring{M}$ of M, such that $C \cap \mathbb{R} \neq \phi$. Then we can write $C \cap \mathbb{R} = (\alpha, \beta)$ where $-1/4 \leqslant \alpha < \beta < 2$. For all $\lambda \in (\alpha, \beta)$, T_λ has a unique attractive k-cycle $\{x_1(\lambda), x_2(\lambda), ..., x_k(\lambda)\}$. This cycle is real and depends continuously on $\lambda \in (\alpha, \beta)$. Conversely if $\{x_1(\lambda), x_2(\lambda), ..., x_k(\lambda)\}$ is a real attractive k-cycle then there is a component C of $\mathring{M}$ with $C \cap \mathbb{R} \neq \phi$ such that $\lambda \in C$. (3) The derivative $(d/dx)\, T_\lambda^k(x)|_{x_1(\lambda)}$ is strictly decreasing in $\lambda \in (\alpha, \beta)$, with value $+1$ at α and -1 at β. At some point $\nu \in (\alpha, \beta)$, called the center of C, $(d/dx)\, T_\lambda^k(x)|_{x_1(\lambda)} = 0$, and then the k-cycle is said to be superstable. (4) $\lambda = \alpha$ is a bifurcation point for the real solution $x_1(\lambda)$ of $T_\lambda^k(x) - x = 0$ of one of two types. Either (i) there is, or (ii) there is not another component C' of $\mathring{M}$ with $C' \cap \mathbb{R} \neq \phi$ and $\alpha \in \overline{C'}$. In case (i) C is said to derive from C' by bifurcation at α. This means that C' is associated with a real attractive p-cycle $\{y_1(\lambda), y_2(\lambda), ..., y_p(\lambda)\}$, where p divides k and $p \neq k$, such that $\{y_1(\nu), y_2(\nu), ..., y_p(\nu)\} = \{x_1(\nu), x_2(\nu), ..., x_k(\nu)\}$. In case (ii) we will say (α, β) is a base. (5) There exists a component C'' of $\mathring{M}$ such that $C'' \cap C = \phi$, $C'' \cap \mathbb{R} \neq \phi$, and $\beta \in \overline{C''}$. Thus, β is a bifurcation point of type (ii).

Further information about bifurcations of real solutions of $T_\lambda^k(x) - x = 0$ now follows upon combining the above information with.[25] The only mechanism by which a real k-cycle of T_λ can appear for increasing λ is via either (i) a pitchfork bifurcation, or (ii) a tangent bifurcation. These correspond exactly to the two types of bifurcation point mentioned above; however, in the description of (i), it must be that k is even and $p = k/2$. Let α be a bifurcation point of type (ii). Then there is an associated real k-cycle $\{x_1(\lambda), x_2(\lambda), ..., x_k(\lambda)\}$ for $\lambda > \alpha$, continuously dependent on λ, which does not have a real continuation for $\lambda < \alpha$. This cycle can be continued to $\lambda = \alpha$, and $\{x_1(\alpha), x_2(\alpha), ..., x_k(\alpha)\}$ is also a real k-cycle. For $\lambda \geqslant \alpha$ there exists a real k-cycle $\{\tilde{x}_1(\lambda), \tilde{x}_2(\lambda), ..., \tilde{x}_k(\lambda)\}$, continuously dependent upon λ, distinct from $\{x_1(\lambda), ..., x_k(\lambda)\}$ for $\lambda > \alpha$ and such that $x_j(\alpha) = \tilde{x}_j(\alpha)$ for $j \in \{1, 2, ..., k\}$.

One of the two k-cycles is attractive and the other is repulsive for $\lambda \in (\alpha, \alpha + \varepsilon)$ for some $\varepsilon > 0$.

From the point of view of the complex plane we see that a tangent bifurcation occurs with increasing λ when two distinct k-cycles, one the complex conjugate of the other, with nonzero imaginary parts, become real at $\lambda = \alpha$ to form a single real k-cycle. As λ increases from $\lambda = \alpha$ to $\lambda > \alpha$ the coalesced pairs of points separate, yielding two real k-cycles one of which is attractive and the other repulsive. On the other hand, a pitchfork bifurcation takes place when a self-conjugate k-cycle, with nonzero imaginary parts, becomes real at $\lambda = \alpha$. In this case, the members of the cycle coalesce in pairs on the real axis to become at $\lambda = \alpha$ an indifferent real $(k/2)$-cycle. Not only at this value of λ does the cycle merge with itself, but also it coalesces with a second real $(k/2)$-cycle which was real and stable for $\lambda \in (\alpha - \varepsilon, \alpha)$ for some $\varepsilon > 0$. When λ is increased from $\lambda = \alpha$ to $\lambda > \alpha$ the self-conjugate k-cycle becomes an attractive real k-cycle, and the $(k/2)$-cycle which was already on the real axis becomes unstable.

We return now to the context of Theorem 8, and consider what happens at ν.

Theorem 9. Let $(\tilde{b}(e_1, e_2, \ldots, e_k)) = \{x_1(\lambda), x_2(\lambda), \ldots, x_k(\lambda)\}$ with $\lambda \in (\nu, \infty)$ be the real k-cycle exhibited in Theorem 8, and let $x_1(\lambda) < x_j(\lambda)$ for $j \neq 1$. Let $\tilde{e}_i = e_i$ for $i \neq k$, $\tilde{e}_k = -e_k$, and if the k-cycle possesses a partner, denote it by $(\tilde{b}(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_k)) = \{\tilde{x}_1(\lambda), \tilde{x}_2(\lambda), \ldots, \tilde{x}_k(\lambda)\}$. Assume that $e_1 e_2 \cdots e_k = -1$. Then $\{x_1(\lambda), x_2(\lambda), \ldots, x_k(\lambda)\}$ is superstable at ν with $x_1(\nu) = 0$, and it possesses a unique real continuation to some largest interval (α, β), containing ν, throughout which it is an attractive real k-cycle. For $\lambda \in (\alpha, \nu)$ it obeys the functional equations

$$\begin{aligned}\tilde{u}_j(x_j(\lambda), \lambda) &= x_j(\lambda) \\ &\quad - (\lambda + \tilde{e}_j \sqrt{(\lambda + \cdots + \tilde{e}_k \sqrt{(\lambda + \tilde{e}_1 \sqrt{(\lambda + \cdots + \tilde{e}_{j-2} \sqrt{x_j(\lambda)}) \cdots})}}) \\ &= 0, \qquad j \in \{1, 2, 3, \ldots, k\}\end{aligned}$$

If the k-cycle is a loner then $\lambda = \alpha$ is a bifurcation point of type (i) at which the cycle takes part in a pitchfork bifurcation. The $(k/2)$-cycle out of which the real k-cycle appears, and which is attractive over some interval immediately preceding the bifurcation point is $(\tilde{b}(e_1, e_2, \ldots, e_{k/2}))$, and $e_1 e_2 \cdots e_{k/2} = -1$. If $\{x_1(\lambda), x_2(\lambda), \ldots, x_k(\lambda)\}$ possesses a partner $\{\tilde{x}_1(\lambda), \tilde{x}_2(\lambda), \ldots, \tilde{x}_k(\lambda)\}$, then the partner obeys $\tilde{u}_j(\tilde{x}_j(\lambda), \lambda) = 0, j \in \{1, 2, \ldots, k\}$, and is a repulsive k-cycle for all $\lambda \in (\alpha, \infty)$. In this case $\lambda = \alpha$ is a bifurcation point of type (ii) at which the k-cycle and its partner coalesce in a tangent bifurcation. If $e_1 e_2 \cdots e_k = +1$ then the k-cycle $(\tilde{b}(e_1, e_2, \ldots, e_k))$ possesses a partner, and the roles of $\{x_1(\lambda), x_2(\lambda),..., x_k(\lambda)\}$ and $\{\tilde{x}_1(\lambda), \tilde{x}_2(\lambda),..., \tilde{x}_k(\lambda)\}$ are interchanged.

Proof. Since $e_1 e_2 \cdots e_k = -1$, we cannot have

$$1 - \frac{e_1 e_2 \cdots e_k}{2^k [x_1(v)\, x_2(v) \cdots x_k(v)]^{1/2}} = 0$$

and the only possibility for stopping the continuation of the real solutions $x_j(\lambda)$ of $u_j(x_j(\lambda), \lambda) = 0$ through decreasing λ-values is $x_1(v) = 0$, which means that the cycle is superstable and v is the center of a component of $\mathring{M}$. It follows that the cycle possesses a unique continuation to some largest open interval (α, β), containing v, throughout which it is an attractive k-cycle. Since the order of visitation is independent of $\lambda \in (\alpha, \infty)$, it follows from Theorem 7A that the cycle must obey either $u_j(x_j(\lambda), \lambda) = 0$ or $\tilde{u}_j(x_j(\lambda), \lambda) = 0$ at each $\lambda \in (\alpha, v)$. But the former is not possible, because if it was true then the k-cycle could be continued through decreasing λ-values to a second center, which contradicts the information about the Mandelbrot given earlier.

Notice that because $\tilde{e}_1 \tilde{e}_2 \cdots \tilde{e}_k = +1$, the real solution $\{x_1(\lambda), x_2(\lambda),..., x_k(\lambda)\}$ of $\tilde{u}_j(x_j(\lambda), \lambda) = 0$ can be continued through decreasing values of λ until $\lambda = \alpha$ at which we have the bifurcation condition

$$\frac{\partial \tilde{u}_j}{\partial x_j}(x_j, \lambda)\bigg|_{\lambda = \alpha} = 1 - \frac{\tilde{e}_1 \tilde{e}_2 \cdots \tilde{e}_k}{2^k [x_1(\alpha)\, x_2(\alpha) \cdots x_k(\alpha)]^{1/2}} = 0$$

For $\lambda \in (\alpha, \alpha + \varepsilon)$ where $\varepsilon > 0$ is sufficiently small the equations $\tilde{u}_j(x_j, \lambda) = 0$, $j \in \{1, 2,..., k\}$, must possess a second distinct solution which we denote by $\{\tilde{x}_1(\lambda), \tilde{x}_2(\lambda),..., \tilde{x}_k(\lambda)\}$. This must be either a $(k/2)$-cycle or a k-cycle according as the bifurcation point of a type (i) or (ii), respectively.

Since for $\lambda \in (\alpha, \alpha + \varepsilon)$ the k-cycle $\{x_1(\lambda), x_2(\lambda),..., x_k(\lambda)\}$ is attractive, all other cycles must belong to B_λ, including in particular $\{\tilde{x}_1(\lambda),..., \tilde{x}_k(\lambda)\}$. It now follows that the latter cycle is given by $(\tilde{b}(\tilde{e}_1 \tilde{e}_2,..., \tilde{e}_k))$ which is a k-cycle if and only if $(\tilde{e}_1, \tilde{e}_2,..., \tilde{e}_k)$ is a k-cycle in Ω, which is to say that $\{x_1(\lambda), x_2(\lambda),..., x_k(\lambda)\}$ possesses a partner. The only other possibility is that α is of type (i) and $(\tilde{b}(\tilde{e}_1, \tilde{e}_2,..., \tilde{e}_k))$ would have to be a $k/2$-cycle, namely, $(\tilde{b}(\tilde{e}_1, \tilde{e}_2,..., \tilde{e}_{k/2})) = (\tilde{b}(e_1, e_2,..., e_k))$, and $\{x_1(\lambda), x_2(\lambda),..., x_k(\lambda)\}$ must be a loner. Since the real $(k/2)$-cycle, out of which the real k-cycle $\{x_1, (\lambda), x_2(\lambda),..., x_k(\lambda)\}$ appeared, must itself be attractive for λ just less than α, we must have $e_1 e_2 \cdots e_{k/2} = -1$. ∎

Proof of Theorem 7(B). It is clear from the above that the order of visitation for a real k-cycle is the same as that for its partner, if it has one. ∎

Example 4. Consider the 4-cycle $(\tilde{b}(-+--))$, for which $x_1 = \tilde{b}(-+--)$ is the least element. Since $(-+-+)$ is not a $\tilde{4}$cycle in Ω, $(\tilde{b}(-+--))$ is a loner and must have appeared by pitchfork bifurcation from the 2-cycle $(\tilde{b}(-+))$. This 2-cycle is itself a loner and must have appeared by pitchford bifurcation from the 1-cycle $(\tilde{b}(-))$. The latter has the partner $(\tilde{b}(+))$, with which it appeared by tangent bifurcation. This example, including the orders of visitation, is summarized in Fig. 2. The dotted portions of the curves denote cycles which do not belong to B_λ and consequently are not represented by λ-chains. These cycles can be indicated by the functional equations which they obey. For $\lambda \in (\alpha_1, \nu_1)$ the attractive 1-cycle obeys $x - (\lambda - \sqrt{x}) = 0$, whilst for $\lambda \in (\nu_1, \alpha_2)$ it obeys $x - (\lambda + \sqrt{x}) = 0$. Similarly, Theorem 9 tells us that the attractive 2-cycle $\{x_1, x_2\}$ which exists for $\lambda \in (\alpha_2, \alpha_3)$, obeys $x_1 - (\lambda - \sqrt{(\lambda - \sqrt{x_1})}) = 0$, $x_2 - (\lambda - \sqrt{(\lambda - \sqrt{x_2})}) = 0$ for $\lambda \in (\alpha_2, \nu_2)$, and $x_1 - (\lambda - \sqrt{(\lambda + \sqrt{x_1})}) = 0$, $x_2 - (\lambda + \sqrt{(\lambda - \sqrt{x_2})}) = 0$, for $\lambda \in (\nu_2, \alpha_3)$.

3.4. How $B_\lambda \cap [0, \infty)$ is Fixed by Order of Visitation

This section is mainly about the calculus of itineraries from the point of view of λ-chains. The underlying content is closely related to the results of Refs. 17, 26, 29, 35, 36, and 39, a good account of which is given in Ref. 14.

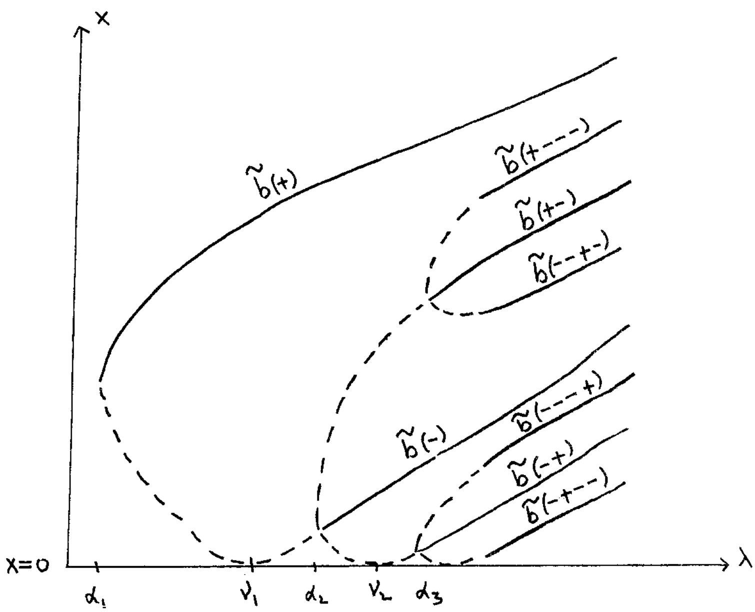


Fig. 2. Sketch of the bifurcation diagram associated with the 4-cycle $[\tilde{b}(-+--)]$, labeled with the corresponding λ-chains. See Example 4.

Our results are obtained more quickly because we are concerned only with quadratic maps and with eventually periodic kneading sequences. Our point of view is different in other ways as well. The calculus of itineraries involves a fairly complicated book-keeping exercise which is handled diagrammatically here, and the output, in λ-chain notation, gives the topology of B_λ in the sense of Section 2.4.

We answer the following question, which concerns $B_\lambda \cap [0, \infty)$ and how it varies with increasing real λ. Let $(\tilde{b}(e_1, e_2, ..., e_k))$ be a real k-cycle. Then what is the full set of other λ-chains which *necessarily* belong to $B_\lambda \cap [0, \infty)$? That is, we continue through decreasing λ-values both the k-cycle and its partner (if it has one) until, at $\lambda = \nu$, one or the other is superstable. Then we wish to determine $B_\nu \cap [0, \infty)$. Equivalently, we define $W_\omega = \{\sigma \in \Omega \mid \tilde{b}(\sigma) \in (0, \infty) \text{ when } \lambda = \nu\}$, so that $B_\nu \cap [0, \infty) = \tilde{b}(W_\omega)$, and we look for W_ω. Notice that if (α, β) is the interval of stability of $(b(\omega))$ (or its partner) then $\tilde{b}^{-1}(B_\lambda \cap [0, \infty))$ is independent of $\lambda \in (\alpha, \beta)$. Also $\tilde{b}^{-1}(B_\lambda \cap [0, \infty))$ is an increasing set-valued function of $\lambda \in \mathbb{R}$.

We describe how to determine W_ω, starting from $\omega \in \Omega$. From ω we obtain the order of visitation of $(\tilde{b}(\omega))$ according to Theorem 7. This order is the same both for the cycle and its partner, if it has one; consequently, it is unnecessary to decide whether it is the cycle or its partner which is superstable at $\lambda = \nu$. The outcome of the computation is the same in any case. Indeed, instead of starting with ω we could begin with the attendant order of visitation.

From the order of visitation implied by ω we can fix certain facts about the real mapping T_ν: $\mathbb{R} \to \mathbb{R}$. Let the real superstable k-cycle be $\{x_1, x_2, ..., x_k\}$, where $x_1 = 0$. Then the set of points $\{(x_1, x_2), (x_2, x_3), ..., (x_{k-1}, x_k), (x_k, x_1)\}$ must lie on the graph of $T_\nu(x) = (x - \nu)^2$. The graph is a parabola with its minimum on the x-axis at x_k; see Fig. 3. Since we do not know ν in general we cannot draw the graph accurately; however, we can make a *sketch graph* which contains the information we need. To do this we mark on both the x and y axes the set of points $\{x_1, x_2, ..., x_k\}$ according to their real order $x_1 < x_{\sigma(2)} < x_{\sigma(3)} < \cdots < x_{\sigma(k)}$. We label the intervals defined by these points with the notation $I_0 = (-\infty, x_1)$, $I_1 = (x_1, x_{\sigma(2)})$, $I_2 = (x_{\sigma(2)}, x_{\sigma(3)}), ..., I_{k-1} = (x_{\sigma(k-1)}, x_2)$ and $I_k = (x_2, \infty)$. We also locate the points whose *coordinates* are $(x_1, x_2), (x_2, x_3), ..., (x_{k-1}, x_k)$ and (x_k, x_1). The sketch graph is completed by joining the neighboring pairs of these points by straight lines, and including both a monotone decreasing straight line through $(x_1, x_{\sigma(2)})$ for $x \in I_0$ and a monotone increasing straight line through $(x_{\sigma(k-1)}, x_2)$ for $x \in I_k$.

Example 5. To illustrate the procedure so far we construct the *sketch graph* for $\omega = (+---)$. Denoting the corresponding 4-cycle $(b(+---))$ by

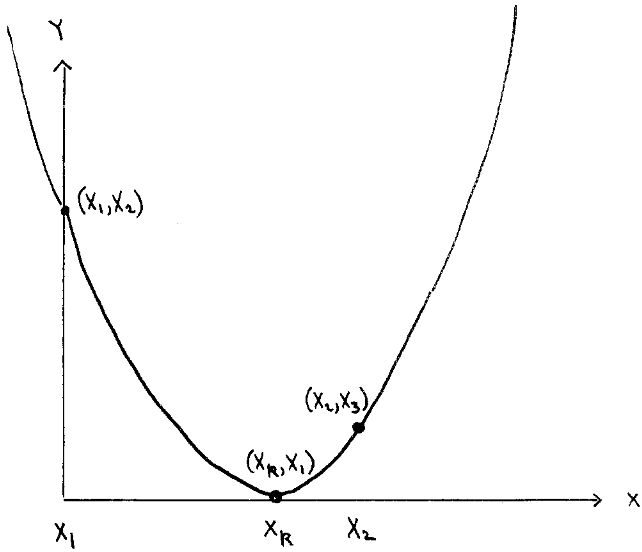


Fig. 3. The graph of $T_v(x)$, when $\{x_1, x_2, ..., x_k\}$ is a superstable k-cycle.

$\{x_1, x_2, x_3, x_4\}$, its order of visitation is $x_1 < x_3 < x_4 < x_2$, and the intervals are $I_0 = (-\infty, x_1)$, $I_1 = (x_1, x_3)$, $I_2 = (x_3, x_4)$, $I_3 = (x_4, x_2)$, and $I_4 = (x_2, \infty)$. The sketch graph is shown in Fig. 4.

Next we use the sketch graph to locate where lie the preimages under T_v of the intervals $I_0, I_1, ..., I_k$. Let $\tilde{R}_\pm$ denote the two branches of the real inverse of T_v, where the negative axis cut is chosen, and

$$\tilde{R}_+(x) = x_k + \sqrt{x}, \qquad \tilde{R}_-(x) = x_k - \sqrt{x}$$

Then for $j \in \{1, 2, ..., k\}$, $\tilde{R}_+(I_j)$ [respectively, $\tilde{R}_-(I_j)$] is contained in exactly one of the intervals $\{I_0, I_1, ..., I_k\}$ which we denote by I_{j+} (respectively I_{j-}). To find I_{j+} (respectively, I_{j-}) one reads off the interval on the x axis which lies to the right (respectively, left) of x_k for which the corresponding portion of the graph has I_j for its set of y values. Notice that $I_{k+} = I_k$ and $I_{k-} = I_0$, whilst I_0 has no real preimages. Once $I_{j\pm}$ have been found for $j \in \{1, 2, ..., k\}$ we construct what we will call the *code* for ω, as follows. Label $k+1$ columns $I_0, I_1, ..., I_k$ and for each $j \in \{1, 2, ..., k\}$ draw two arrows, one labelled $\oplus$ from the column I_{j+} to the column I_j, and one labeled $\ominus$ from the column I_{j-} to the column I_j. In place of using columns one can use points if k is not too large.

Example 5 (continued). From Fig. 4, we readily find $\tilde{R}_-(I_1) \subset I_2$, whence $I_{1-} = I_2$, and $\tilde{R}_+(I_1) \subset I_3$, whence $I_{1+} = I_3$. Similarly $I_{2-} = I_2$,

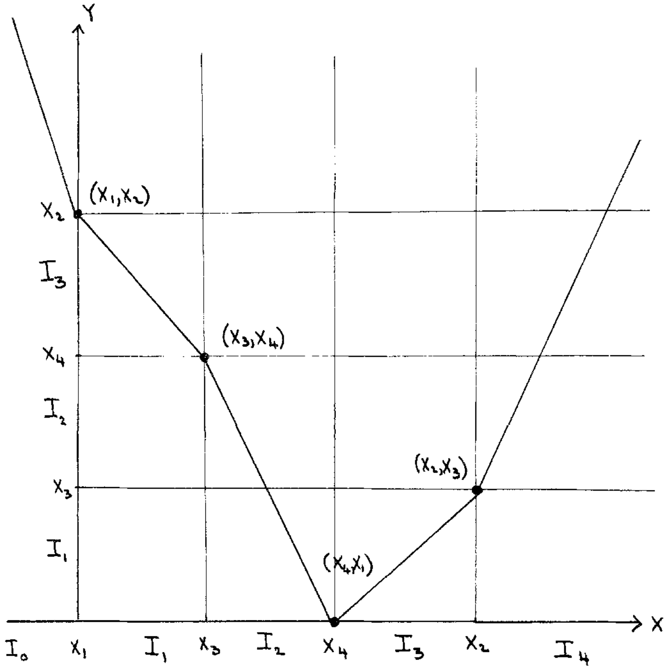


Fig. 4. The sketch graph for $\omega = (+---)$. It is convenient to mark the intervals I_0, I_1, I_2, I_3, and I_4 on both axes.

$I_{2+} = I_4$, $I_{3-} = I_1$, $I_{3+} = I_4$, $I_{4-} = I_0$, and $I_{4+} = I_4$. This information is represented by the code in Fig. 5a, and also by the code in Fig. 5b where points are used instead of columns.

In general, once the code for ω has been found, the set W_ω can be calculated as follows. To find a member of W_ω we simply write down from left to right any half infinite sequence of plus and minus signs which is encountered upon following arrows from column to column (or point to point) commencing at I_j with $j \neq 0$. The set of all elements of Ω which are obtainable in this manner is exactly W_ω, and the set of corresponding negative axis λ-chains is $B_\nu \cap \gamma$, as stated formally in the next theorem.

Theorem 10. Let ω belong to a k-cycle in Ω. Then W_ω is equal to the set of elements of Ω which are obtained by following arrows in the code for ω, as described above.

Proof. Let the continuation through decreasing values of λ of either $(\tilde{b}(\omega))$ or its partner be superstable at $\lambda = \nu$. Then we recall that W_ω is the set of $\sigma \in \Omega$ such that $b(\sigma) \in (0, \infty)$ when $\lambda = \nu$.

(a)

(b)

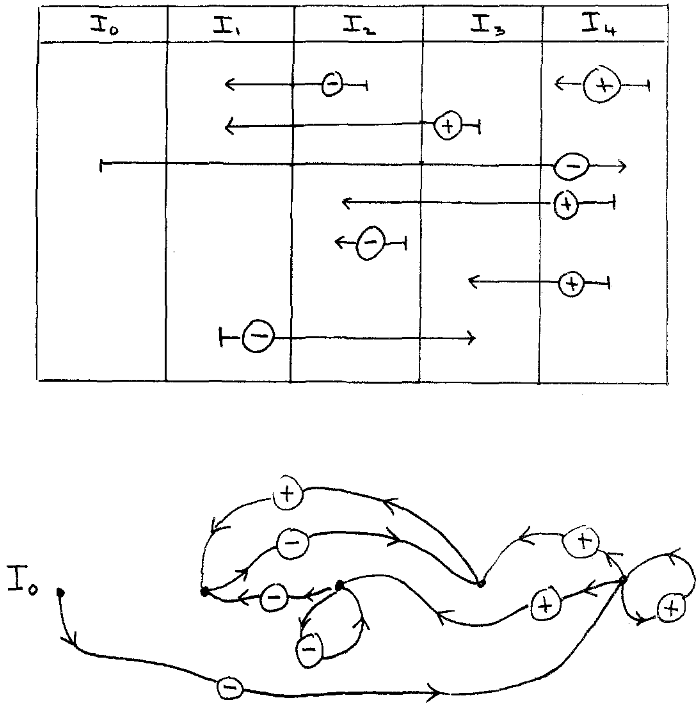


Fig. 5. (a) The code for $\omega = (-+++)$. (b) The code for $\omega = (-+++)$ using points in place of columns.

Let $\sigma \in \Omega$ be obtained by following arrows in the code for ω, as described above. Then fix $\lambda = \nu$ and consider the sequence of inverse functions $\{\tilde{R}_\sigma^n(x)\}_{n=1}^{\infty}$ as defined prior to Theorem 2. There must exist two intervals I_j and I_m with $j, m \in \{1, 2,..., k\}$, and an infinite subsequence $\{\tilde{R}_\sigma^{n(i)}(x)\}_{i=1}^{\infty}$ such that

$$\tilde{R}_\sigma^{n(i)}(I_m) \subset I_j \qquad \text{for} \quad i \in \{1, 2,...\}$$

(The interval I_j is the one at which commences the chain of arrows giving σ, and I_m is an interval which is visited infinitely many times on following that chain.) Since the k-cycle $(\tilde{b}(\omega))$ or its partner is superstable the set of critical points for all branches of $\tilde{R}^n(z)$ for any n is contained in the k-cycle and does not belong to I_m. Hence the sequence of functions $\{\tilde{R}_\sigma^{n(i)}(z)\}$ is holomorphic in a neighborhood of I_m and hence by Ref. 13, Theorem 6.2, possesses a subsequence which converges uniformly on closed subintervals of I_m to an element of B_ν. This element of B_ν lies in $I_j \subset (0, \infty)$. Note that since the sequence of sets $\{\tilde{R}_{\sigma}^{n(i)}(I_m)\}_{i=1}^{\infty}$ is nested and decreasing, the convergent subsequence can be taken to be $\{\tilde{R}_{\sigma}^{n(i)}(z)\}_{i=1}^{\infty}$, $z \in I_m$. The element to which the subsequence converges can be uniquely represented $\tilde{b}(\tilde{\sigma})$ for some $\tilde{\sigma} \in \Omega$, since every real element of B_v is given by exactly one negative axis λ-chain. Noting that $T_v^n(\tilde{b}(\tilde{\sigma}))$ must lie to the left or right of $x_k = \lambda = v$ according as $\sigma_n = -1$ or $+1$, respectively, we conclude that $\sigma = \tilde{\sigma}$, and hence $\sigma \in W_\omega$ as desired.

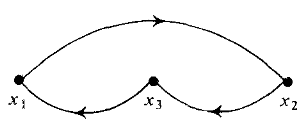
Conversely, let $\xi \in W_\omega$. Then $x = \tilde{b}(\xi) \in B_v \cap [0, \infty)$ and for any positive integer n there exists $y \in B_v \cap [0, \infty)$, namely, $y = T_v^n(\tilde{b}(\xi))$, such that

$$x = \tilde{R}_{\xi}^{n}(y)$$

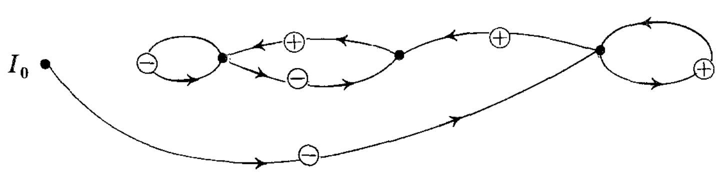
But this says that from the code for ω by following arrows we must be able to find the sequence of signs belonging to the first n components of ξ. We conclude that ξ can be obtained by following arrows in the code for ω. ∎

Example 5 (continued). From the code for $\omega = (+---)$ shown in Figs 5a and 5b we read off at once that W_ω must contain the two 1-cycles $(+)$ and $(-)$, and also the 2-cycle $(+-)$. In fact in this case we see that W_ω consists of all elements of Ω expressible in the form $(+++\cdots+\}---\cdots-\}+-)$. This notation extends that which was introduced earlier, and means all elements of the form $(+)$, or $(+++\cdots+\}-)$ where there are finitely many initial plus signs, or $(+++\cdots+--\cdots-\}+-)$ where there are finitely many plus signs at the beginning, following by finitely many minus signs, followed finally by $+-+-+-+-\cdots$ Notice in this example that W_ω does not contain any elements which are not eventually periodic, in contrast to the next example.

Example 6. We take $\omega = (+--)$, for which the order of visitation is



The corresponding code is



Notice that although this code is simpler to write down than the one for $(+---)$, it implies that W_ω is much larger than for the previous example. W_ω now consists of all elements of Ω which begin with finitely many plus signs, followed by any sequence of signs from $\{-, -+\}$. In particular we see that the existence of the real 3-cycle $(\tilde{b}(-++))$ implies that the Julia set contains real k-cycles of all other orders. Notice that the implied cycles are by no means arbitrary, and their orders of visitation are implicit. For example, the real 4-cycle $(\tilde{b}(-+++))$ is implied but $(\tilde{b}(-++-))$ is not. If $(\tilde{b}(-++-))$ is a real 4-cycle then the real 3-cycle $(\tilde{b}(-++))$ is implied. Thus, although from the more general setting of Sharkovskii[39] and of Li and Yorke[29] we have that "real period 3 implies real period 4," we have in our more specialized situation that "period 4 with a certain order of visitation implies period 3," and so on.

We remark that if, in determining the elements of W_ω from the code for ω, we allow the sequence of arrows to start of I_0 was well as at $I_1, I_2, \ldots, I_k$, then we obtain the subset of Ω for which the negative axis λ-chains lie on the real axis; that is, we include those members of B_λ which lie on the negative real axis.

We summarize as follows. Any $\omega \in \Omega$, or order of visitation of a real k-cycle, fixes a sketch graph and in turn a code. The code fixes W_ω and hence $B_\nu \cap [0, \infty)$. Once $B_\nu \cap [0, \infty)$ is known the equivalence class structure for positive axis λ-chains is implied. The latter fixes not only the orders of visitations of all those cycles which necessarily coexist with the original one, but also the topology of B_ν as described in Section 2.4.

4. THE FIRST CASCADE VIEWED FROM THE COMPLEX PLANE

4.1. Description of the First Cascade

Here we illustrate aspects of the preceding theory with a description of the progression of Julia sets which occur when λ increases and $T_\lambda(x) = (x-\lambda)^2$ yields its first cascade of period doubling bifurcations.

The cascade corresponds to the successive occurrence of the sequence of real 2^n-cycles

$(\tilde{b}(-))$	2^0-cycle
$(\tilde{b}(-+))$	2^1-cycle
$(\tilde{b}(-+--))$	2^2-cycle
$(\tilde{b}(-+---+-+))$	2^3-cycle
$(\tilde{b}(-+---+-+-+---+--))$	2^4-cycle

Let the 2^nth cycle in this cascade be $(\tilde{b}(\omega_{2^n}))$. Then $(\omega_{2^{n+1}})$is obtained by repeating (ω_{2^n}) twice and then reversing the last sign. One can readily check that (ω_{2^n}) corresponds to the smallest element of $(\tilde{b}(\omega_{2^n}))$, and also that except for the 1-cycle each cycle in the cascade is a loner, derived by pitchfork biforcation from its predecessor.

Let $(b(\omega_{2^n}))$ be superstable at $\lambda = \nu_n$. Then it follows from Theorem 9 that the ν_n's obey the sequence of functional equations

$$f_0(\nu_0) := \nu_0 = 0$$
$$f_1(\nu_1) := \nu_1 - \sqrt{\nu_1} = 0$$
$$f_2(\nu_2) := \nu_2 - \sqrt{(\nu_2 + \sqrt{(\nu_2 - \sqrt{\nu_2}))}} = 0$$
$$f_3(\nu_3) := \nu_3 - \sqrt{(\nu_3 + \sqrt{(\nu_3 - \sqrt{(\nu_3 - \sqrt{(\nu_3 - \sqrt{(\nu_3 + \sqrt{(\nu_3 - \sqrt{\nu_3}))))))}}}}}} = 0$$

Here the negative axis branch cut is understood. These functions can be used as the basis for numerical methods for the calculation of the "universal constants" such as $\lim_{n \to \infty} \nu_n = \nu_\infty$ (equal to the Myrberg[34] number 1.401155...) and

$$\lim_{n \to \infty} \frac{\nu_n - \nu_{n-1}}{\nu_{n+1} - \nu_n}$$

(equal to the Feigenbaum[23] ratio 4.66920...). We note that the analysis of the λ-chains involved allows one to focus attention on exactly the k-cycles one wishes to study without having to consider irrelevant cycles of the same order.

The conclusion of the first cascade is marked by the occurrence of the "∞-cycle" $(\tilde{b}(-+---+-+-+---+--\cdots))$ referred to by Ref. 20 and by Ref. 14.

With the aid of Theorem 10 we calculate

$$W_{\omega_1} = \{(+)\}$$
$$W_{\omega_2} = \{(+++\cdots+\}-)\}$$
$$W_{\omega_4} = \{(+++\cdots+\}---\cdots-\}+-)\}$$

and

$$W_{\omega_8} = \{(+++\cdots+\}---\cdots-\}+-+-\cdots+-\}+---)\}$$

The notation is that introduced in Example 5.

In Figs. 6a–6f we represent the successive structures of B_λ. We use the notation x for the real 1-cycle $(\tilde{b}(-))$, and we write $T_\lambda^{-1}(x) = \{x, y\}$ where $x \neq y$. We use • to denote elements of k-cycles and ▯ to denote first

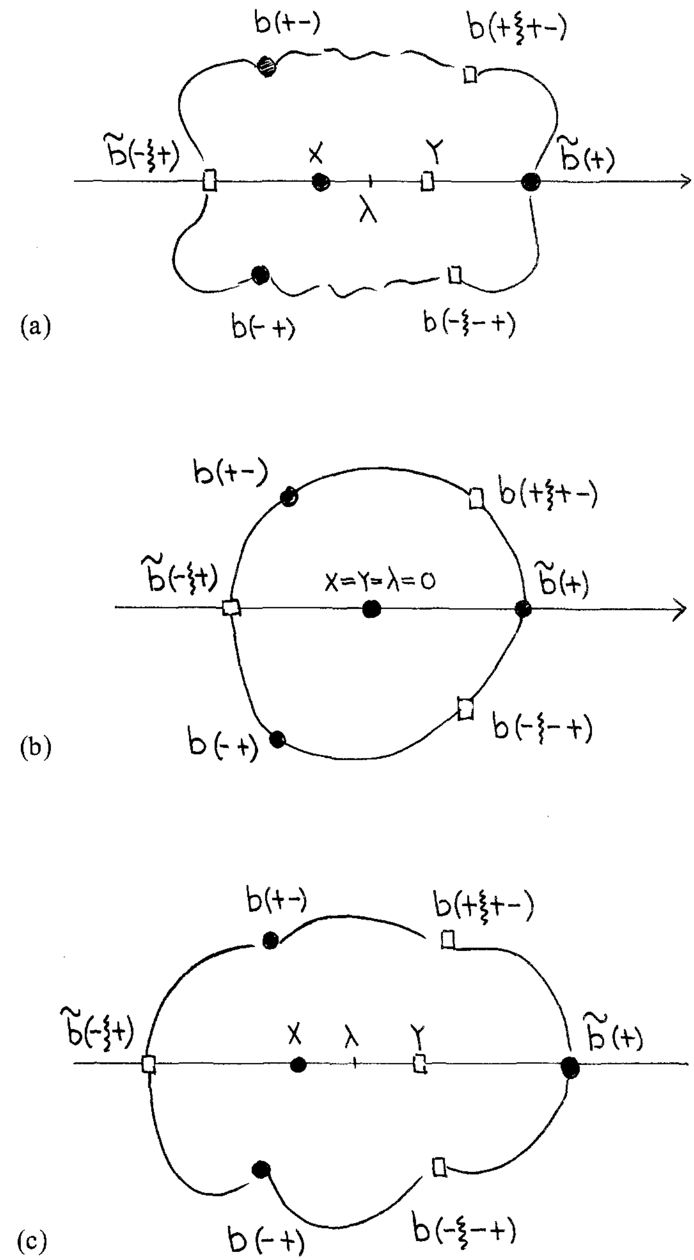


Fig. 6. (a) Schematic representation of B_λ when $-1/4 = \lambda_0 < \lambda < \nu_0 = 0$. B_λ is a simple Jordan curve. (b) Sketch of B_λ when $\lambda = \nu_0 = 0$. B_0 is the unit circle. (c) Schematic representation of B_λ when $0 = \nu_0 < \lambda < \lambda_1$. B_λ is a simple Jordan curve.

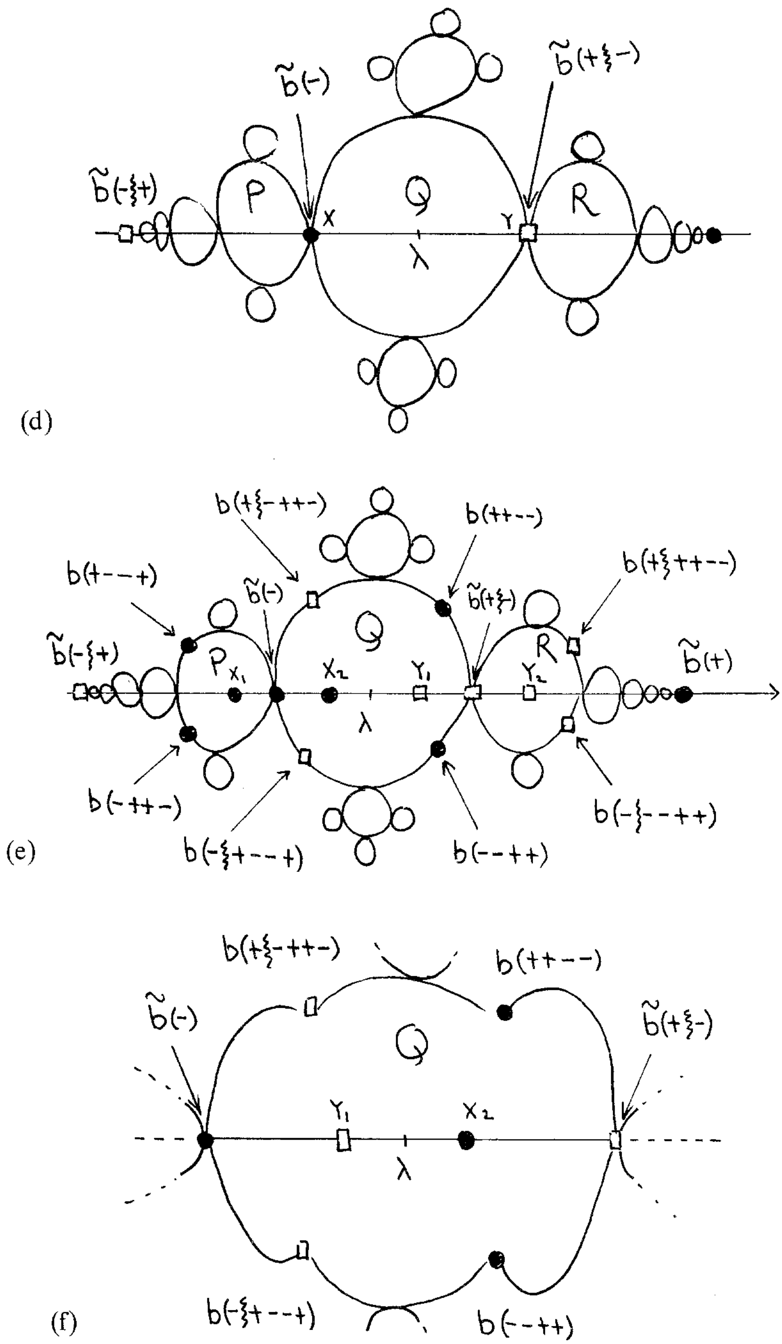


Fig. 6. (d) Schematic representation of B_λ when $\lambda = \lambda_1$, at which occurs the first pitchfork bifurcation. (e) Schematic representation of B_λ when $\lambda_1 < \lambda < \nu_1$. T_λ now possesses an attractive 2-cycle. (f) Schematic representation of the continuation of the component labeled Q in Fig. 6e. Now $\nu_1 < \lambda < \lambda_2$. Compare with Fig. 6c.

predecessors of elements of k-cycles which do not themselves belong to k-cycles. The critical point λ is shown in each figure. The λ-value at which the real 2^n-cycle $(\tilde{b}(\omega_{2^n}))$ first appears will be denoted λ_n.

In Fig. 6a we give a schematic representation of B_λ in the complex plane, when $-1/4 = \lambda_0 < \lambda < \nu_0 = 0$. In this case B_λ is a simple Jordan curve which separates $\mathbb{C}$ into two components, the immediate attractive sets of x and ∞. The only elements of Ω whose equivalence classes consist of more than one element are $(+) \sim (-)$, $(+\}-) \sim (-\}+)$ and $(s_1 s_2 \cdots s_n + \}-) \sim (s_1 s_2 \cdots s_n - \}+)$, where each $s_i \in \{+,-\}$. The only real members of B_λ are the repulsive 1-cycle $b(+) = b(-) = \tilde{b}(+)$ and its real preimage $b(+\}-) = b(-\}+) = \tilde{b}(-\}+)$. x and y obey the functional equations $x = \lambda + \sqrt{x}$ and $y = \lambda - \sqrt{x}$. Also shown is the 2-cycle $\{b(+-), b(-+)\}$ together with its first predecessor $\{b(+\}+-), b(-\}-+)\}$. The continuation of the 2-cycle when it first becomes real, will coalesce with real 1-cycle which we denote by $(\tilde{b}(-))$ to yield the first pitchfork bifurcation in the cascade.

In Fig. 6b $\lambda = \nu_0 = 0$ and the attractive 1-cycle $x = (\tilde{b}(-))$ is superstable. In Fig. 6c $\nu_0 < \lambda < \lambda_1$, and the main difference from the situation in Fig. 6a is that the real ordering of x and y is reversed, and for all $\lambda > \nu_0$ we have $x = \lambda - \sqrt{x}$ and $y = \lambda + \sqrt{x}$. As λ increases from ν_0 to λ_1 the complex 2-cycle $\{b(+-), b(-+)\}$ approaches the real attractive 1-cycle x. As it does so the preimage $\{b(-\}+-), b(+\}+-)\}$ approaches the preimage y of the 1-cycle. Not shown are the higher-order preimages of the 1-cycle and the 2-cycle involved: the predecessors of order n of the 2-cycle lie on the simple Jordan curve B_λ and can be separated into pairs each of which approaches one of the predecessors of order n of x. The latter all lie in the bounded component of the complement of B_λ.

In Fig. d $\lambda = \lambda_1$. B_λ is now pinched together at x, where $b(+-) = b(-+) = (\tilde{b}(-))$. Prior to λ_1 $\tilde{b}(-)$ actually yields the 2-cycle, which has nonzero imaginary parts, whilst for $\lambda > \lambda_1$ it gives a real 1-cycle. Similarly B_λ is pinched together at y where $b(+\}+-) = b(-\}-+) = \tilde{b}(+\}-)$. The other multiple points in the figure represent a few of the countable infinity of other "pinchpoints" at which preimages of higher order of x and of the 2-cycle are coincident. The 1-cycle x is now indifferent rather than attractive, and lies on B_λ. The meanings of the chains indicated in Fig. 6d are clear from continuation. The three components in the figure which are labeled P, Q, and R are related by $T_{\lambda_1} P = Q$, $T_{\lambda_1} Q = P$, and $T_{\lambda_1} R = Q$.

In Fig. 6e B_λ is represented for $\lambda_1 < \lambda < \nu_1$. In this case there is an attractive 2-cycle, namely, $\{x_1, x_2\} = (\tilde{b}(-+))$, and the previously attractive 1-cycle denoted by $(\tilde{b}(-))$ now belongs to B_λ. $\{x_1, x_2\}$ has emerged from x, leaving the λ-chain $\tilde{b}(-)$ on B_λ, whilst the preimage $\{y_1, y_2\}$ of the 2-cycle has emerged from y, leaving $\tilde{b}(+\}-)$ on B_λ. Thus, the 2-cycle which was earlier on B_λ has left it to become an attractive 2-cycle, its stability having been transferred from the previously attractive 1-cycle which has now rejoined B_λ.

In Fig. 6e we also show the 4-cycle $\{b(+--+), b(--++), b(-++-), b(++--)\}$ and its first predecessor $\{b(-\}+--+), b(-\}--++), b(+\}-++-), b(+\}++--)\}$, which will be involved in the next bifurcation.

The situation for $\nu_1 < \lambda < \lambda_2$ is essentially the same as for $\lambda_1 < \lambda < \nu_1$ except that the real ordering of x_2 and y_1 is interchanged. In Fig. 6f we represent for $\nu_1 < \lambda < \lambda_2$ the continuation of the component labeled Q in Fig. 6e. In Figs. 6e and 6f, P and Q denote the two components of the immediate attractive set of $\{x_1, x_2\}$. The behavior of the boundary of Q, as λ increases from λ_1, is similar to that of the whole of B_λ as λ increases from λ_0. Indeed, if we consider T_λ^2 in place of T_λ we see that $\{b(++--), b(--++)\}$ becomes a 2-cycle instead of part of a 4-cycle, whilst x_2 becomes an attractive 1-cycle. As λ increases $\{b(++--), b(--++)\}$ pinches inwards to join x_2, whilst its predecessors under T_λ^2 on the boundary of Q move to coincidence with the predecessors under T_λ^2 of x_2 in Q. Similar deformations take place with regard to P and to the countable infinity of other components of the attractive set of $\{x_1, x_2\}$.

Let ∂P, ∂Q, and ∂R denote, respectively, the boundaries of P, Q, and R in Figs. 6e and 6f. One readily shows that ∂P is given by the set of positive axis λ-chains $b(s_1 s_1' s_2 s_2' s_3 s_3' \cdots s_n s_n' \cdots)$ where each $s_i \in \{+, -\}$ and s_i' is the opposite sign to s_i. Similarly ∂Q is given by $b(s_0 s_1 s_1' s_2 s_2' s_3 s_3' \cdots s_n s_n' \cdots)$ and ∂R is given by $b(s_0 s_0 s_1 s_1' s_2 s_2' \cdots s_n s_n' \cdots)$. Each of these boundaries is a simple Jordan curve because the Julia set is hyperbolic. The boundaries of the countable infinity of other components of the attractive set of the 2-cycle are obtained by taking inverse images of all orders of ∂P, and the set of positive axis λ-chains of which a given one of these boundaries consists can be deduced from the successive branches of the inverse T_λ which when applied to ∂P yield the desired boundary. Note that, for $\lambda_1 < \lambda < \lambda_2$, B_λ is the closure of the set of all Jordan curves thus obtained. Similar observations apply with regard to the boundaries of the immediate attractive sets which occur as the cascade proceeds.

We can now make some deductions about the locations of cycles during the cascade. Note first that the cycles which participate in the cascade, prior to their becoming real, are given by the sequence of positive axis λ-chains

(2^0) $\{b(+)\}$ and $\{b(-)\}$

(2^1) $\{b(+-), b(-+)\}$

(2^2) $\{b(+--+), b(--++), b(-++-), b(++--)\}$

(2^3) $\{b(+--+-++-), b(--+-++-+), \cdots\}$

$\vdots$

When the 2^n-cycle in this sequence becomes real, and $(\tilde{b}(\omega_{2^n}))$ is attractive, then for all $j \in \{1, 2, 3, ...\}$ the 2^{n+j}-cycle in the sequence resides upon the boundary of the immediate attractive set of $(\tilde{b}(\omega_{2^n}))$. For example, all of the sequence starting with $\{b(+-), b(-+)\}$ lie upon B_λ when $(\tilde{b}(-))$ is attractive, and all of the sequence starting with $\{b(+--+), ...\}$ are located upon $\partial P \cup \partial Q$ when $(\tilde{b}(-+))$ is attractive. The general assertion can be proved inductively.

Our second deduction concerns cycles which do *not* lie either on the boundary of the immediate attractive set of $(\tilde{b}(\omega_{2^n}))$ when this cycle is attractive, or on any of the Jordan curves which are finite-order preimages of the boundary of the immediate attractive set. We have already illustrated how one can calculate the set of positive axis λ-chains which make up the boundary of the immediate attractive set of a cycle. Clearly any k-cycle whose positive axis λ-chain is not included must itself not lie in the boundary of any component of the complement of B_λ which does not contain infinity. For example, when $\lambda_1 < \lambda < \lambda_2$ the 3-cycle $\{b(++-), b(+-+), b(-++)\}$ does not lie upon the boundary of any of the "bubbles" in Fig. 6e and 6f, because its λ-chains are not included in the ones, described above, which make up these boundaries. Thus the 3-cycle occurs only as an accumulation point of the boundaries in Figs. 6e and 6f. Similarly we discover that when the 3-cycle $(\tilde{b}(++-))$ is attractive, none of the cycles involved in the first cascade are located upon the boundary of the immediate attractive set of the 3-cycle or any of its finite-order preimages.

4.2. The Feigenbaum Functional Equation

In this section we relate, mainly formally, the preceding description to the theory of the Cvitanović–Feigenbaum–Coulett–Tresser functional equation. In so doing we make a number of conjectures, concerning the existence and nature of various limits.

We begin by describing a renormalization procedure, associated with the first cascade, along the lines suggested by Refs. 16, 23 and others, but viewed here in terms of the Julia set. As explained in Section 4.1, during the first cascade B_λ undergoes a sequence of structural changes which have an approximately self-similar feature. The Julia sets B_{ν_n} associated with the superstable 2^n-cycles, $n = 0, 1, 2, ...$, look something like the sketches in Fig. 7.

It is seen that the transformation which takes B_{ν_0} to B_{ν_1} is approximately repeated in going from B_{ν_1} to B_{ν_2} in the sense that each "bubble" in B_{ν_1} is replaced by an appropriately scaled approximate copy of B_{ν_1}. Similarly, in going from B_{ν_2} to B_{ν_3}, each "bubble" in B_{ν_2} is replaced by an appropriately scaled approximate copy of B_{ν_1}.

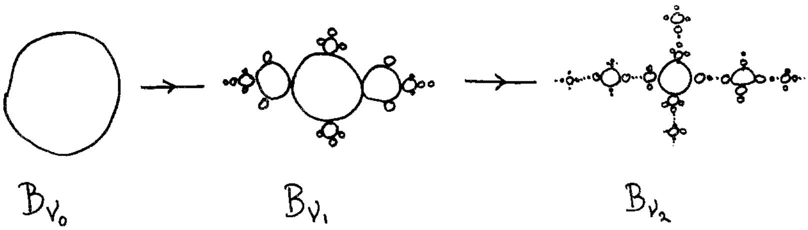


Fig. 7. Sketches of successive superstable Julia sets.

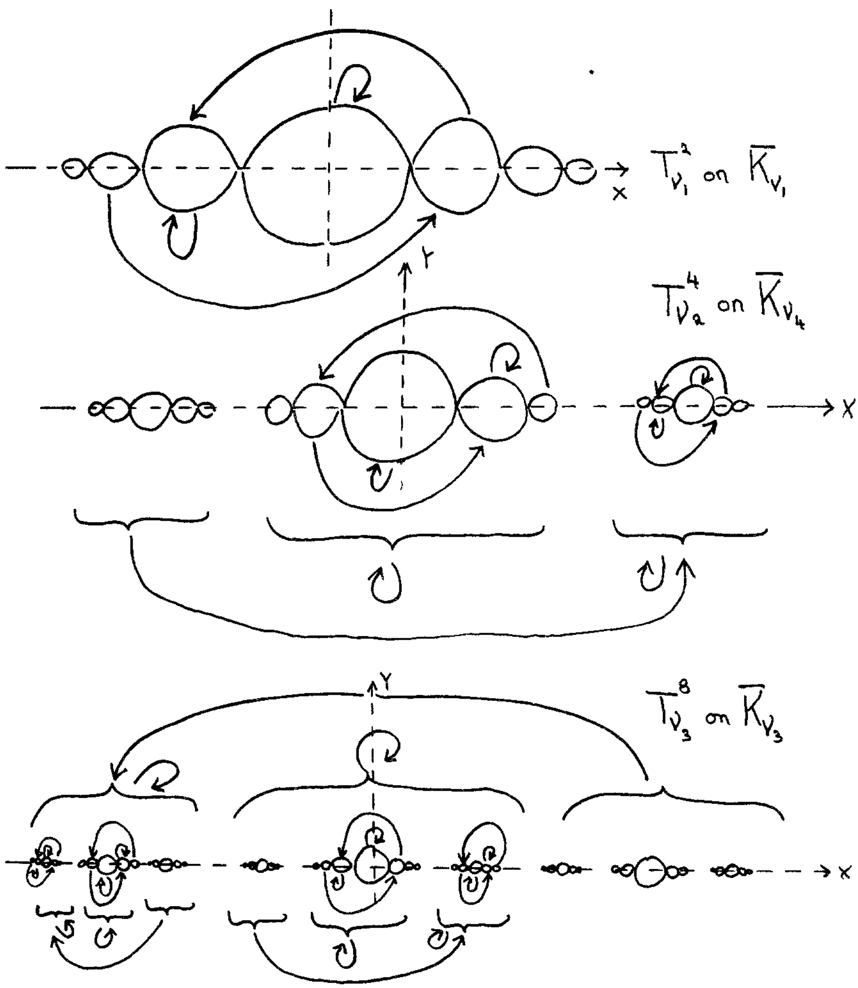


Fig. 8. Schematic representation of the dynamics of $T_{\nu_n}^{2n}$ on K_{ν_n}, $n = 1, 2, 3, \ldots$.

For this section we take

$$T_\lambda(z) = z^2 - \lambda$$

The Julia set is the same as for $(z-\lambda)^2$, but shifted to the left by λ. B_λ is now symmetrical about the origin.

Let K_{ν_n} denote the complement of B_{ν_n}, minus the component which contains ∞, and let $\overline{K_{\nu_n}} = B_{\nu_n} \cup K_{\nu_n}$. Then we can relate the dynamics of $T_{\nu_n}^{2^n}$ on $\overline{K_{\nu_n}}$ to the dynamics of $T_{\nu_{n+1}}^{2^{n+1}}$ on $\overline{K_{\nu_{n+1}}}$. We convey this relation by implication, with the aid of Fig. 8. The sets represent $\overline{K_{\nu_1}}$, $\overline{K_{\nu_2}}$, $\overline{K_{\nu_3}}$,... and the arrows show which component is mapped into which under $T_{\nu_1}^2$, $T_{\nu_2}^4$, $T_{\nu_3}^8$,..., respectively. Note the successive reversals.

We interpret universal scaling here as saying that the middle San Marco structure[31] of B_{ν_n} is related to that $B_{\nu_{n+1}}$ by a constant scaling factor Λ in the limit as $n \to \infty$. Hence we assume that a limiting structure

$$J = \lim_{n\to\infty} \Lambda^{-n} K_{\nu_n}$$

is well defined, and looks something like the sketch in Fig. 9. Correspondingly we also assume that a limiting mapping ϕ, analytic on J, is obtained by defining

$$\phi(z) = \lim_{n\to\infty} (-1)^n \Lambda^{-n} T_{\nu_n}^{2^n}(\Lambda^n z)$$

(the minus sign takes care of the successive reversals). The manner in which ϕ maps the components of J^0 can readily be deduced from Fig. 8. We call $J \backslash J^0$ the "Julia set" of ϕ.

It should be possible to calculate the action of ϕ on J precisely. Good insight into the character of ϕ, and support for the above assumptions, is obtained by restricting attention to the action of $T_{\nu_n}^{2^n}$ on the central component P_n of K_{ν_n}. Here we show how this action is conformally conjugate via an intertwining map E_n, to $T_{\nu_{n+1}}^{2^{n+1}}$ acting on P_{n+1}, with the aid of a corresponding pair of Böttcher functional equations.

We derive these latter equations here from an electrical viewpoint. Let ∂P_n be the boundary of P_n. Let $g_n(z)$ denote the Green's function for P_n with pole at $z=0$. Then

$$g_n(T_{\nu_n}^{2^n}(z)) = 2g_n(z) + \pi n i \qquad \text{for} \quad z \in P_n \backslash \{0\}$$

because $T_{\nu_n}^{2^n}(P_n) = P_n$ and $T_{\nu_n}^{2^n}(\partial P_n) = \partial P_n$ so both sides vanish as z approaches ∂P_n, and because $g_n(z)$ can be written as $\log(1/z)$ plus a regular function whilst $T_{\nu_n}^{2^n}(z) = (-1)^n c^n z^2 + O(z^3)$ where c_n is a positive constant.

Let $f_n(z)$ be the unique conformal mapping which takes P_n onto P_0, with $f_n(0) = 0$ and $f_n(z) > 0$ when $z > 0$. Then we must have for $z \in P_n \backslash \{0\}$

$$g_n(z) = g_0(f_n(z)) = \log[1/f_n(z)]$$

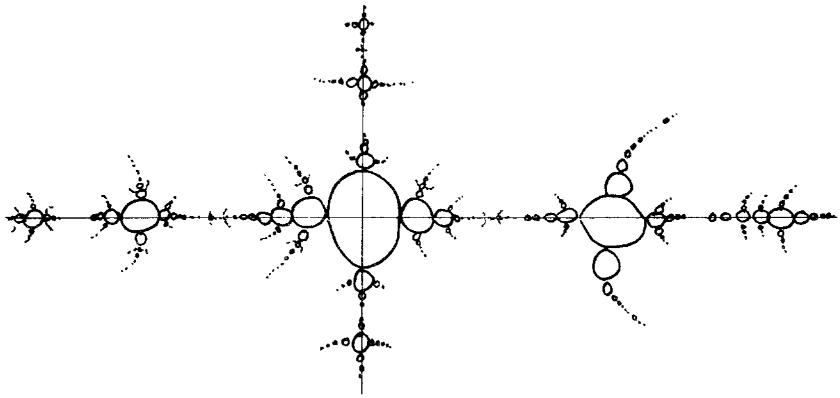


Fig. 9. Sketch of the infinite structure J. It goes to infinity along both axes.

where we have used the fact that P_0 is the disk $\{z \in \mathbb{C} \mid |z| < 1\}$ and $g_0(z) = \log(1/z)$. It follows that

$$f_n(z) = \exp[-g_n(z)] \qquad \text{for} \quad z \in P_n$$

and we now have

$$f_n(T_{v_n}^{2^n}(z)) = (-1)^n (f_n(z))^2 \qquad \text{for} \quad z \in P_n$$

which is the Böttcher functional equation associated with the superstable fixed point $z = 0$ of $T^{2^n}(z)$. We set

$$E_n(z) = f_{n+1}^{-1}(f_n(z))$$

Then for $z \in P_{n+1}$ we have

$$E_n(T_{v_n}^{2^n}(E_n^{-1}(z))) = -T_{v_{n+1}}^{2^{n+1}}(z)$$

which connects the action of $T_{v_n}^{2^n}$ on P_n with that of $T_{v_{n+1}}^{2^{n+1}}$ on P_{n+1}. Our earlier scaling assumptions are equivalent here to supposing that $E_n(z)$ converges uniformly and rapidly to Λz on the central component P of J, and that the limiting map ϕ can be analytically continued out of P.

We next consider the relation between ϕ, the Feigenbaum function, and some results of Lanford.[30] Define, with domain the set of functions even and analytic in a neighborhood of the real interval $[-1, 1]$, a renormalization operator F by

$$(F\Psi)(z) = -a^{-1}\Psi \circ \Psi(az)$$

where a is a certain positive constant. We review briefly some results of Lanford concerning F—for precision see Ref. 30. There is a function $\tilde{g}(z)$ in the domain of F, such that $\tilde{g}(0)=1$ and $\tilde{g}(1)=-a$, whose restriction to $[-1, 1]$ is a fixed point of F. Associated with the fixed point $\tilde{g}$, F admits locally invariant local stable and local unstable manifolds, of codimension one and dimension one, respectively. Let $\tilde{W}_s$ and $\tilde{W}_u$ denote some particular local stable and unstable manifolds. For some integer j the family of functions $F^j(\tilde{T}_\lambda(z))$, with $T_\lambda(z)=1-\lambda z^2$, as a curve in function space, parametrized by λ, crosses $\tilde{W}_s$ transversally at $\lambda=\lambda_\infty$ (a number near 1.401155).

The relation between Lanford's normalization and that appropriate to our setting is obtained by noting that

$$\tilde{T}_\lambda(z)=S\circ T_\lambda\circ S^{-1}(z)$$

where $S(z)=-z/\lambda$. The form of F remains the same, but its domain is scaled and in our frame the Feigenbaum function is $g(z)=-\lambda\tilde{g}(-z/\lambda)$. We denote the corresponding manifolds by W_s and W_u.

We believe, and assume in what follows, that $\phi\in W_u$, $\lambda_\infty=\nu_\infty$, and $a=\Lambda$. This seems entirely consistent with the description of how to find W_u given by Feigenbaum.[24] The point is that $T_{\nu_\infty}\in W_s$ so that

$$\lim_{n\to\infty} F^n(T_{\nu_\infty}(z))=g(z)$$

whereas we have considered

$$\lim_{n\to\infty} F^n(T_{\nu_n}(z))=\phi(z)$$

Let us consider the "Julia set" for $g(z)$. It is easily verified that the Julia set for the polynomial $F^n(T_{\nu_\infty}(z))$ is $B_{\nu_\infty}^{(n)}=(1/a^n)B_{\nu_\infty}$. Now, because T_{ν_∞} possesses no finite attractive or indifferent k-cycles, B_{ν_∞} is treelike (see Ref. 3, for example) and can be constructed as follows. Let

$$I_0=[-\tfrac{1}{2}-(\nu_\infty+\tfrac{1}{4})^{1/2}, \tfrac{1}{2}+(\nu_\infty+\tfrac{1}{4})^{1/2}]$$

and let

$$I_n=\bigcup_{j=0}^{n} T_{\nu_\infty}^{-j}(I_0)$$

then $\{I_n\}_{n=0}^{\infty}$ is an increasing family of trees, with

$$B_{v_\infty} = \lim_{n\to\infty} I_n$$

$B_{v_\infty}^{(n)}$ consists of the same object magnified by $(1/a)^n$.

We conjecture that the sequence of magnified trees

$$\{(1/a^n) I_{n+m}\}_{n=0}^{\infty}$$

converges to a treelike object G_m, and that $\{G_m\}_{m=0}^{\infty}$ is an increasing family of sets. Then we define

$$G = \lim_{m\to\infty} G_m$$

to be the "Julia set" of g.

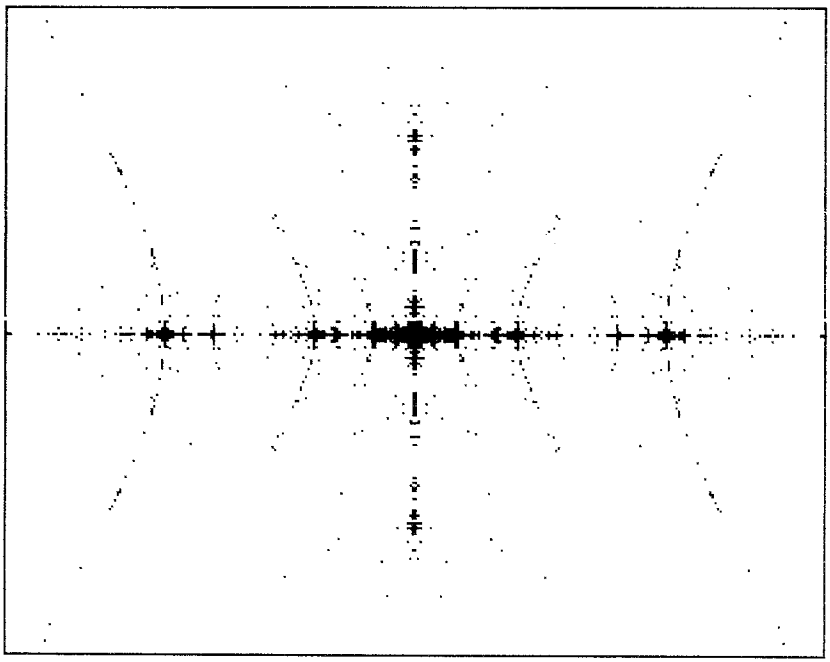


Fig. 10. The heart of the Julia set for $z^2 - v_\infty$. This represents part of B_{v_∞}, contained in a box centered at the origin of width and height 4.852×10^{-2}. It was obtained on a Terak microcomputer, single precision, with a screen of 320×240 pixels. The black dots represent points which after 400 iterations under T_λ $(\lambda = 1.40115)$ have magnitude less than 100. Suitably scaled, the locations where some of the major limbs cross the axes are approximately the same as corresponding locations in the Epstein–Lascoux tree.

We claim that G_0 is the Epstein–Lascoux "tree" $g^{-1}(\mathbb{R})$ which is partially illustrated in Ref. 22. This claim, which is formally reasonable, is also supported by numerical results: we have calculated approximate pictures of $B_{v_\infty}^{(n)}$, $n = 0, 1, 2, 3, 4, 5$, and, in addition to noting apparent convergence within a central window, we have found that there are clearly demarked major limbs, branching off the x and y axes, whose locations appear to converge to corresponding locations in the Epstein–Lascoux tree. See Fig. 10.

As yet we know little about G; some numerical results suggest the possibility $G = \mathbb{C}$. If this is true then the action of g on $\mathbb{C}$ may be thought of as being very close to that of a high-degree polynomial upon its Julia set. It would also suggest that the two-dimensional Lebesgue measure of B_{v_∞} is nonzero.

Finally, we connect the above conjectures back to ϕ. If $\phi \in W_u$ then it follows from Lanford's results that at least a subsequence of inverses $\phi_n \in F^{-n}(\phi)$ converges to g. The "Julia sets" of these inverses would be the boundaries of $J_n = a^n J$, each made of infinitely many bubbles but such that in the limit all bubbles would be scaled out of existence, producing the set G. This would certainly include the real and imaginary axes, and if we are right, much, much more.

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